LOCATION OF RAILROAD CURVATURES

VAN DE GRAAFF

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ON

THE LOCATION

OF

RAILROAD CURVATURES;

BEING AN INVESTIGATION OF ALL THE PRINCIPAL FORMULAS
WHICH ARE REQUIRED FOR FIELD OPERATIONS, IN
LAYING CURVES AND TANGENT LINES
TO PASS THROUGH GIVEN POINTS.

BY

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NEW-YORK:
PUBLISHED AT THE OFFICE OF THE RAILROAD JOURNAL
1836.

Entered according to Act of Congress, in the year 1836, by D. K. Minor, in the Clerks Office of the District Court of the southern District of New-York.

ON THE LOCATION OF RAILROAD CURVATURES; BEING AN INVESTIGATION OF ALL THE PRINCIPAL FORMULAS WHICH ARE REQUIRED FOR FIELD OPERATIONS, IN LAYING CURVES AND TANGENT LINES, TO PASS THROUGH GIVEN POINTS.

Art. 1.—When it becomes necessary for a road to pass from a right line into a curve, the former should in all cases be laid tangent to the latter. It is, however, unnecessary, in ordinary roads, to adhere to this principle with the rigor of an accurate geometrical construction; but with reference to a railroad, it should be observed with the greatest care and precision, *as well as the general selection and distribution of the various curvatures.* This is an obvious consequence of the well known law of *inertia*; for any two points being fixed, in the general direction of a route, through which it is proposed to lay a line of railroad composed of several curves and intervening tangents, the cost of construction is in most cases not the only requisite datum to fix the definite location of the intermediate points.

There are, indeed, very few varieties of ground, except in a very broken country, which will not admit of several different lines, connecting the same points, all at nearly an equal expense of construction, and all within the same *limits* of curvature. It therefore becomes an object to make such a selection of right lines and curves, and such a distribution thereof, as will produce, *at a given expense*, the most efficient road.

The effect of curvature, upon the *cost of transportation*, does not fall within the present subject; and the only general rules which can, therefore, be here given, as a guide for the arrangement of a line, are the following: Endeavor to open the closer

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curves by increasing their length, or by increasing the length of some of those which have a greater radius, or by giving some tangent line a different direction. But a sufficient dexterity in the judicious arrangement of the curvatures of a line can only be had from experience ; and the principal object which is here proposed is an investigation of the various formulas which are required in facilitating the operations in the field.

2. In order to obtain results which are of commodious application in the field, it is necessary to use only circular arcs for curves ; but when the situation of the ground is such as to render a continuous arc of the same curvature inapplicable to the purposes required, a curvilinear line may nevertheless be traced in such a manner as to take any desired form, in order to pass through any number of designated points, by merely connecting together a succession of circular arcs, having different *radii of curvatures*. To make a good selection in the field, for those different circular arcs, requires skill and judgment ; but the only additional care to be observed in the operations of tracing such a selected line, will arise from the necessity of connecting the different arcs in such a manner as that they may have a common normal at their point of connection.

The following well-known properties of the circle exhibit the first principles which will be necessary in the subsequent inquiries.

Theo. I. The angle of deflection* between two equal and contiguous chords is measured by the arc which is subtended by

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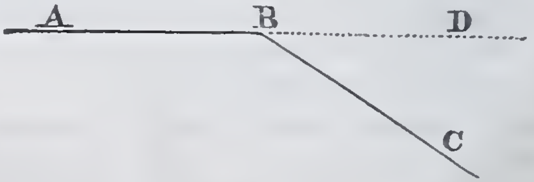
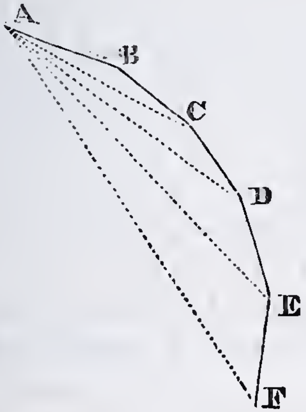
* The anale of deflection between two lines which meet in a point, is the angle contained between either line and the other line produced.



Thus DBC is the angle of deflection between the two lines AB and BC.

either of the chords. II. If there be a tangent line, and a chord from the touching point, the angle of deflection from the tangent line into the chord is one-half of the angle of deflection between two such equal chords. III. Tangent lines from the two extremities of any arc, form equal angles with the connecting chord: each equal to one-half of the intercepted arc. IV. Any angle at the circumference, standing upon a given chord, will have a constant value for all positions of that chord upon the circumference; and this constant value is one-half of the arc subtended by the given chord. These principles are presumed to be familiar to the reader; and have been here transcribed only for the sake of convenient reference.

3. Let any number of equal straight lines, AB, BC, CD, &c.,



be connected together in such a manner at their extremities, that the angles BAC, CAD, DAE, &c., may be all equal to each other; it then follows from Theo. IV., Art. 2, that the points A, B, C, D, &c., will all be situated in the circumference of the circle. And this obviously suggests the common method of tracing a circular arc in the field, by means of a chain whose length is AB, or BC, or CD, &c., and an instrument for measuring the equal angles BAC, CAD, DAE, &c. It is thus easy to set as many stations in a curve as can be seen from an instrument placed at the station A. The *position* of the first chain, AB, is here taken arbitrarily, and does not effect the *curvature.*

4. In using the formulas which will be deduced in the course of the present inquiry, it must be remembered that all measurements of *distance* are supsosed to be made in chains, and the decimal parts of a chain. The chain will therefore be the *unity* of length, and may have any value whatever; but as this will be a constant quantity in the field, it follows, that the *curvature* of a line, traced as in the last article, can only be made variable by assuming, in succession, a different value for those equal angles BAC, CAD, DAE, &c. These equal angles will therefore be called the *modulus* of curvature, and will always be denoted by a letter T. The modulus of curvature will therefore be a constant quantity in the same curve, but variable in different curves; and any curve will be given, when its modulus of curvature is known. The letter n will be taken to denote the number of chains composing any arc which may be under consideration, and the letter D to represent the number of degrees contained in that arc. It is then obvious, from Theo. IV., Art. 2, that each chain will subtend a portion of the circumference equal to 2T; and consequently the following formula is the evident result,

$$D = n \times 2T. \qquad \text{(I.)}$$

This expression gives the number of degrees which any arc contains, when the modulus of curvature and number of chains are known.

5. In tracing a circular arc of given curvature, by means of chords of more or less than ane chain in length, the particular modulus of curvature corresponding to any given length of chord will be nearly proportional to the length of that chord, when that length is only a very small part of the whole circumference. For, in this case, the length of the given chord will be nearly proportional to the number of degrees in the arc which it subtends, and therefore, also, by Theo. IV., Art. 2, nearly proportional to its particular modulus of curvature.

Hence T denoting the modulus of curvature for a given curve with a chain whose length is unity, then $p \times T$ will be nearly the modulus of curvature for the *same curve* when traced by means of chords whose length is p chains. This result is only approximative; but a rigorous formula will be investigated hereafter. It is evident, upon the same principles, that (I.) is rigorous only when n is an *integer* number of chains.

6. Since each chain in a curve subtends a portion of the circumference equal to double the modulus of curvature, it follows, from Theo. I., Art. 2, that the angle of deflection between any two contiguous chains will likewise be equal to double the modulus of curvature. And it thus also obviously appears, from Theo. II., Art. 2, that the angle of deflection from any chain, into a tangent line at either of its extremities, will be equal to the modulus of curvature.

7. When a given curve contains more than one chain of the same modulus of curvature, and the direction of a tangent line at any station be required, it should never be found from the direction of the first chain at that station; for small errors will arise, both in placing the instrument vertically over one of the stations, and in taking the back sight to the centre of the other, and the angular effect of these errors will obviously diminish as the two stations are more remote. In order, therefore, to obtain the direction of a tangent line at any station in a given curve, direct the instrument into a chord, embracing a number of chains in the proposed curve, denoted by n; and it then appears from Theo. III., Art. 2, that $\frac{1}{2}$D will express the angle of deflection from the selected chord into the required tangent line. Hence, by Art. 4, the formula required in obtaining the direction of the tangent line is,

$$\tfrac{1}{2}D = n \times T\,. \qquad \text{(II.)}$$

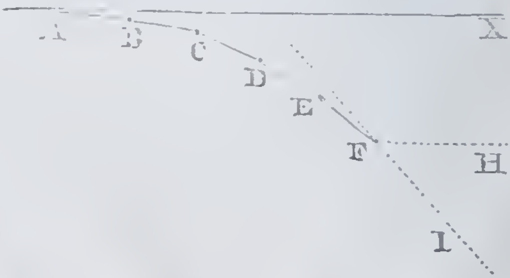
8. The general method of tracing a curve in the field, having the modulus of curvature given, will appear from Art. 3; but it remains yet to show how such a curve may be laid to a given tangent line at a given point. It is obvious from Art. 6, that the required object will be attained by laying the first chain from the given point with an angle of deflection from the given tangent line equal to the modulus of curvature of the proposed

curve. For then the same curve may obviously be continued from that first chain to any number of chains, by the method of Art. 3.

9. It is frequently necessary to pass from one given curve into another, having a different modulus of curvature. When this case occurs, the instrument must be placed at the station where the change of curvature is to be made; and let it be directed into the *tangent line of the first given curve* at that station, agreeably to the method in Art. 7. The instrument may then be considered as adjusted, so as to be in readiness, either to lay a *continuation* of the first given curve, or to lay any other curve from another given modulus of curvature, in such a manner as to have a common tangent with the first curve at the *point* where the two curves pass into each other; and it is sufficiently obvious that, in either of these two cases, the selected curve must be traced agreeably to the principles given in Art. 8.

When, therefore, a change of curvature occurs at any station, the angle of deflection between the two chains at that station will be equal to the *sum of two moduli of curvatures.*

10. Let AB, BC, CD, &c. represent the successsive chains by which a given curve, ABCD, &c., may be traced from its origin at the station A, in such a manner as to touch a given right line, AX, at that point. It is proposed to investigate a formula which will express the inclination of the given tangent line AX, to the nth chain in a given curve, estimating from the origin at the point A.



Take EF to represent the nth chain; and let the letter E denote the required inclination of the line EF to the given tangent line AX. The inclination of the first chain, AB, is expressed by the given modulons of curvature T, agreeably to Art. 6, or 8; and consequently, the inclination of the second chain, BC, will be expressed by T + 2T, or 3T. In like manner the inclination of third chain CD, is expressed by 3T + 2T, or 5T. Hence the inclination of all the chains, AB, BC, CD, &c., will obviously be expressed by T, 3T, 5T, &c., respectively. The problem is therefore evidently reduced to the following very simple inquiry, viz.: To find the value of the nth term of the series 1, 3, 5, 7, &c. But agreeably to the principles of arithmetic, this nth term will be expressed by $2n-1$. The desired formula will therefore be,

$$E = T \times \overline{2n-1}. \qquad \text{(III.)}$$

The expression (III.) will be frequently very convenient in the field when it is required to compute the magnetic bearing of any particular chain in a long curve, when the bearing of the given tangent line at the origin is known. If the country be free from metallic matter, and the instrument be in proper order for use, the magnetic bearing will always serve as a very useful check in detecting any material error which may have been inadvertently committed in the previous deflections. By way of a numerical illustration of this, let the given tangent line AX bear due North; and let the given modulus of curvature be 2°, to find the bearing of the 20th chain. Here $T = 2°$, $n = 20$; and therefore $E = 2° \times \overline{40-1} = 78°$; and if the curve be supposed to bend to the left, then the bearing of the 20th chain is N. 78° W. If the observed course do not correspond with this result, some error has been committed in tracing the curve.— This method of proof is so very simple, that it should he attended to frequently in tracing long curves, in order to avoid useless labor after some error may have been already committed. The magnetic bearing of a particular chain in a curve is also sometimes required in the field for other purposes.

11. Let ABCD, &c., represent a given curve traced as in the last article; and take EF to represent the nth chain estimated from the origin of the point A. It is proposed to find a me-

thod of directing the instrument, when placed at the station F, into a position parrellel to the given tangent line AX at the origin of the curve.

Direct the instrument into the chord, through any of the back stations, as for instance the station C or D, (see fig. Art, 10,) ; and, from that chord, deflect into the tangent FI, at the station F, agreeably to the method given in Art. 7. Take D′ to denote the inclination of the tangent FI, to the given tangent AX at the origin of the curve, and it evidently follows from Art. 6, that, $D' = E + T$; or, substituting for E its value from (III), the following expression is at once obtained,

$$D' = 2nT. \qquad \text{(IV.)}$$

Hence, from the tangent FI, deflect an angle IFH equal to $2nT$; and the instrument will then have the required position FH.

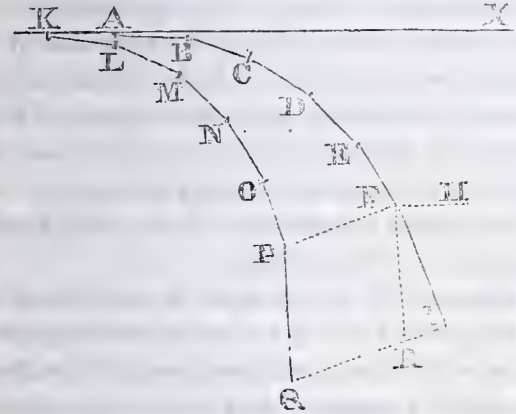
The principle which has just been explained will very frequently be highly useful in the field. For if the origin of the given curve ABCD, &c., be removed from the point A, to any other point in the tangent line AX, then the station F will be changed exactly by the same quantity, and in the same direction, upon the line FH, as will be hereafter shown.

12. Let any two curves be laid upon the same tangent line, and take T and T′ to represent the two moduli of curvatures, and let each curve pass into a tangent at the extremities of the nth and mth chain respectively. It is then proposed to determine a formula which will express the inclination of those latter tangents to each other.

Take z to denote the inclination required. From (IV.) it appears, that $2nT$, and $2mT'$, respectively, will express the inclination of each of the proposed tangents to the common tangent at the origins of the two curves ; and hence the difference between the quantities $2nT$ and $2mT'$ will obviously express the value of z. The formula required is therefore the following,

$$z = 2nT - 2mT'. \qquad \text{(V.)}$$

The expression (V.) will find an application in the field in all cases where two tangents are under consideration ; for in computing the rate at which any two tangents converge or diverge, their inclination to each other is, of course, the first datum required. The principles contained in (V.) and its application to different cases occurring in the field, will be more easily explained by a reference to a figure. Let A B C D, &c., represent the curve whose modulus of curvature is denoted by T, and let EF represent the *n*th chain, and FI the tangent at F. In like manner let KLMN, &c., represent the curve whose modulus of curvature is denoted by T′, and let OP represent the *m*th chain, and PQ the tangent at P.



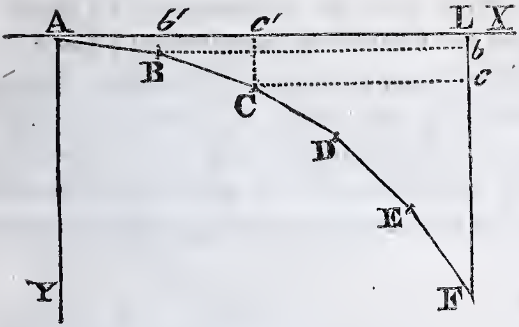
Draw FH parellel to KX, the common tangent at the origins of both curves; and draw, also, FR parallel to the tangent PQ. The inclination of the tangent FI, to the common tangent KX, will evidently be expressed by the angle HFI, or $2nT$; and the inclination of the tangent PQ, to the same common tangent, will in like manner he expressed by the angle HFR, or $2mT'$. Hence the angle IFR, which measures the inclination of the two tangents, FI and PQ, *to each other*, will be expressed by the quantity $2mT'—2nT$. When, therefore, the value of z in (V.) comes out *negative*, it shows that the line FR falls below the tangent FI; that is, *the two tangents in that case diverge*. When z comes out positive, the line FR will be situated above the tangent FI; that is, in such a case the two tangents PQ and FI *will converge.* It is here supposed, however, that the two curves do not intersect each other.

It is sufficiently obvious that the angle $2nT$ will remain the same, whatever position may be given to the origin A in the tangent line AX; and it hence evidently follows, that the distance AK, between the two origins, has no connection whatever with the inclination of the two tangents, FI and PQ, to each other.—The distance AK will, however, have an effect upon the line PF, which expresses the distance between the extremities of the two curves. Supposing the distance AK, and the moduli of curvatures, with the number of chains, n and m, contained in each curve, to be all given, the line PF may be computed by methods which will be hereafter explained; and then the angle IFR, which measures the inclination of the two tangents, FI and PQ, to each other, being known from (V.) it will be easy to compute the situations of any proposed points, in those two tangents which may come under consideration in the field, by the common methods of plane trigonometry.

When the distance PF is very small in comparison with the length of the tangents FI or PQ and when these tangents are but little inclined to each other, every necessary calculation in the field may be made by means of very simple approximative methods. All this will be fully explained in the subsequent articles.

13. Let AB, BC, CD, &c., represent the successive chains by which a given curve ABCD, &c., may be traced from its origin at the station A, in such a manner as to touch a given right line AX, at that station; and let EF represent the nth chain, counted from the origin at A.

Taking AX, and AY, for a system of rectangular co-ordinate axes, it is proposed to investigate formulas which will express the values of the co-ordinates AL and LF, of the station at F.



Put $x = \text{AL}$, and $y = \text{LF}$. The inclination of the first chain AB, to the axis of x, is found by Art. 10 to be $= \text{T}$: and therefore the projections of AB upon the co-ordinate axes will be $\text{A}b' = \text{AB} \cdot \text{Cos. T}$, and $\text{L}b = \text{AB} \cdot \text{Sin. T}$. But as all lines are supposed to be measured in chains, it follows that the projections of AB upon the co-ordinate axes are, $\text{A}b' = \text{Cos. T}$, and $\text{L}b = \text{Sin. T}$. Again, by Art. 10, the inclination of the second chain BC to the axis of x is $= 3\text{T}$, and consequently the projections of BC upon the co-ordinate axes, are $b'c' = \text{Cos. } 3\text{T}$, and $bc = \text{Sin. } 3\text{T}$.

In like manner let each of the other chains be projected upon the co-ordinate axes ; and then taking the sums of those projections, the following equations will obviously be the result,

$$x = \text{Cos. T} + \text{Cos. } 3\text{T} + \text{Cos. } 5\text{T} + \cdot \cdot \cdot \cdot \cdot \cdot \; \text{Cos. } \{\text{T} \cdot \overline{2n-1}\} \cdot$$

$$y = \text{Sin. T} + \text{Sin. } 3\text{T} + \text{Sin. } 5\text{T} + \cdot \cdot \cdot \cdot \cdot \cdot \; \text{Sin. } \{\text{T} \cdot \overline{2n-1}\} .$$

The reason for writing the last term, in each of these two series, as the nth term, requires no explanation ; and the *sum* of n terms of each series being taken agreeably to the known principles of analytical trigonometry, the following formulas will be the result,

$$x = \frac{\text{Sin. } 2n\text{T}}{2 \text{ Sin. T}}$$

$$y = \frac{\text{Sin. }^2 n\text{T}}{\text{Sin. T}} \cdot \qquad \text{(VI.)}$$

The form in which the expressions (VI.) appear is that most convenient for logarithmic computations; and it is also a form which will be required hereafter for other purposes. But for use in the fields, with a table of natural sines and cosines, the expression for the value of y may have a better form. Thus, agreeably to the principles of analytical trigonometry, 2 Sin. $^2 n$T = 1 — Cos. $2n$T; and consequently the expressions (VI.) become,

$$x = \frac{\text{Sin. } 2n\text{T}}{2 \text{ Sin T}}$$

$$y = \frac{1 - \text{Cos. } 2n\text{T}}{2 \text{ Sin T}}. \qquad \text{(VII.)}$$

Such are the formulas which it was proposed to investigate. For an example in figures, let the modulus of curvature be 1°, and suppose it were required to find the value of AL and LF, corresponding with the extremity of the 40th chain. In this case then, T $2n$ = 80°: and by the table of natural sines and cosines, at the end of this volume, I find Sin. 80° = .98481, Cos. 80° = .17365, and Sin. 1° = .01745; hence.

$$x = \frac{.98481}{.03490} = 28.214 \text{ chains; and}$$

$$y = \frac{1 - .17365}{.03490} = \frac{.82635}{.0349} = 23.675 \text{ chains.}$$

If, therefore, the chain used in tracing the curve be 100 feet in length, then AL = 2821 feet, and LF = 2367 feet.

14. Let the modulus of curvature be given from which a circular arc is traced in the field with a given chain; it is then proposed to determine the radius of the arc described.

It is very obvious that when n is made variable in (VII.), the *maximum* value of x will express the required radius; but when x is a *maximum*, it follows that Sin. $2n$T must be a maximum also, if it be supposed that Sin. T remains constant. Now, the quantity Sin. $2n$T obtains its *maximum* value when $2n$T = 90°; in which case Sin. $2n$T = 1. Hence, denoting the radius of the described circle, *in chains*, by R, the result is,

$$R = \frac{1}{2\ \text{Sin. T}}. \qquad \text{(VIII.)}$$

The most simple formula which can possibly be obtained to express the radius of curvature may be had by means of the cosecant of the angle T. For by the principles of trigonometry, Cosec. $T = \frac{1}{\text{Sin. T}}$; and consequently (VIII. becomes,

$$R = \frac{1}{2}\ \text{Cosec. T.} \qquad \text{(IX.)}$$

To save the trouble of computation, the following table is here subjoined:

T	R	T	R	T	R
0°. 3′	572.96	1°. 3′	27.28	2°. 3′	13.98
6′	286.48	6′	26.05	6′	13.65
9′	190.99	9′	24.91	9′	13.33
12′	143.24	12′	23.88	12′	13.03
15′	114.59	15′	22.92	15′	12.74
18′	95.49	18′	22.04	18′	12.46
21′	81.85	21′	21.22	21′	12.19
24′	71.62	24′	20.47	24′	11.94
27′	63.66	27′	19.76	27′	11.70
30′	57.30	30′	19.10	30′	11.46
33′	52.09	33′	18.49	33′	11.24
36′	47.75	36′	17.91	36′	11.02
39′	44.07	39′	17.37	39′	10.81
42′	40.93	42′	16.85	42′	10.61
45′	38.20	45′	16.37	45′	10.42
48′	35.81	48′	15.92	48′	10.24
51′	33.70	51′	15.29	51′	10.06
54′	31.83	54′	15.08	54′	9.88
57′	30.16	57′	14.69	57′	9.72
1°. 0′	28.65	2°. 0′	14.33	3°. 0′	9.55

For an application of the above table to an example, suppose it be required to determine the radius corresponding to a modulus of curvature of 1°, as in the last numerical example. Looking in the column marked R, and opposite to 1° in the column marked T, I find 28.65, which is therefore the radius in chains; and if the modulus of curvature, 1°, appertains to a chain whose length is 100 feet, then the radius of the arc described will be 2865 feet. But if the modulus of curvature, 1°, has reference to a chain whose length is only 50 feet, then the curve traced will have a radius of only the half of 2865 feet.

15. It appears from (VIII.) that the radius of curvature is directly as the length of the chain, and inversely as the sine of the modulus of curvature. Hence, if the radius of curvature be given, then the length of the chain will be directly proportional to the sine of the modulus of curvature. If, therefore, a curve be traced from a given modulus T, and with a chain whose length is unity, and it be required to find what modulus will trace the same curve with a *chord* whose length in chains is p, let the required modulus be denoted by T′, and we have the proportion $1 : p :: \text{Sin. } T : \text{Sin. } T'$; from which is obtained the formula,

$$\text{Sin. } T' = p \times \text{Sin. } T. \qquad \text{(X.)}$$

Example. Suppose the length of the chain to be 100 feet, and let it be required to determine the modulus of curvature which will trace a curve whose radius is 10743 feet, by taking chords whose lengths are 10 chains each. Here the radius of curvature is 107.43 chains; and by inspecting the table of radii, given in the last article, I find that the radius 107.43 is situated between the moduli, 0° 15′, and 0° 18′: and therefore, by proportional parts, I find the true modulus which corresponds to the radius 107.43 to be 0° 16′, and which would, consequently, trace the proposed curve with the given chain.

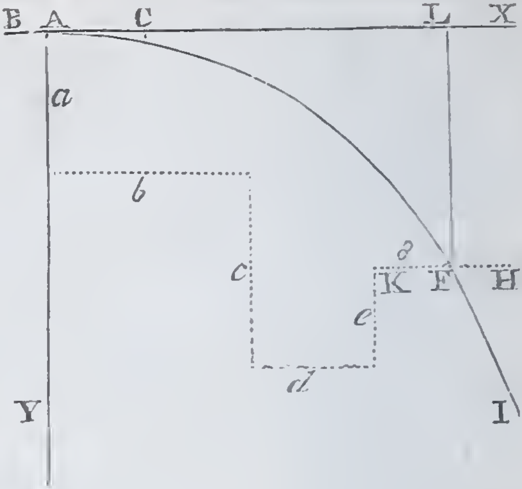
Hence, $\text{Sin. } T' = 10 \times \text{Sin. } 0° 16' = 10 \times .00465 = .0465$; and by the table of natural sines, I therefore find $T' = 2° 40'$, which is the modulus required.

In all cases where the angle T′ does not exceed two or three degrees, the result obtained from (X.) will not differ, by any material quantity, from that obtained agreeably to the principle given in Article 5.

16. Take a system of rectangular co-ordinate axes, having their origin at a given station in a tangent line, from which a certain required curve is to be laid, passing through a point designated by the co-ordinates x, y; the given tangent line coinciding with the axis of x. Parallel respectively to each of the co-ordinate axes, let any number of rectangular lines be traced from the origin, and terminating in the point designated for the required curve to meet; these rectangular lines being selected in any convenient manner to pass any obstacle which may happen to occur. Let the algebraic sums of each of those rectangular lines be taken, agreeably to the axis to which they are respectively parallel. Those sums will obviously give the values of the co-ordinates x, y; and from thence it is proposed to determine a formula, expressing the value of the modulus of curvature of the required curve. And it is also required, from the same data, to determine a method by means of which the instrument may be immediately directed into the true tangent at the designated point x, y.

Let AX represent the given tangent line, A the given origin, and AX, AY, the co-ordinate axes. Take F for the given point, designated by the co-ordinates $AL = x$, and $LF = y$; and let the lines marked with the small letters, a, b, c, d, &c., denote the rectangular lines, originating at A, and terminating at F, and traced in any arbitrary manner parallel respectively to each co-ordinate axis. The algebraic sums of those lines give

$$x = b + d + f$$
$$y = a + c - e$$



It will in all cases be more convenient to determine the values of x and y, agreeably to the method here proposed, than to trace the lines AL and LF in a direct manner, even if the latter were always practicable. Very often, however, the three, a, b, c, will be sufficient.

The values of the co-ordinates x, y, having now been determined, the next object is to find the necessary modulus of curvature which will trace a curve from the origin at A, passing the point F. For this purpose let each of the expressions (VII.) be squared,* and let the last of the two be divided by the sum of the squares; the result is,

$$\text{Sin. T} = \frac{y}{x^2 + y^2}. \qquad \text{(XI.)}$$

The formula thus obtained is remarkably simple, and convenient for use; and it may perhaps be well to observe, that the *safest* method of recording the lines a, b, c, &c., in the field, will be to take on a piece of paper the form

$$\left\{\begin{matrix} x = & \qquad \\ y = & \qquad \end{matrix}\right\},$$

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[*] See note, Art. 20.

and then record each line in its proper equation, and with its proper *sign*, immediately as their values are determined. Example: A system of rectangular lines having been traced to a designated point, let the resulting equations be

$$\left\{ \begin{array}{l} x = 20 - 2 + 4 \\ y = 8 + 3 + 1. \end{array} \right\}$$

In this case, then, $x = 22$ chains, and $y = 12$ chains, and

$$\text{Sin. T} = \frac{12}{484+144} = \frac{12}{628} = .01911\,;$$

and by the table of natural sines I therefore find, $\text{T} = 1^\circ\, 5\frac{3}{4}'$; which modulus will trace the curve required.

But if the curve AF, as first determined by means of (XI.) should appear to be too abrupt, or if from any other cause it should be considered advisable to commence the curve from a different origin upon the tangent line AX, as for instance at B or C, and still retain the same point F, then the necessary modulus of curvature from the new origin is easily obtained from the former measurements. Eor taking $\propto$ to denote the distance AB or AC, the new co-ordinates will obviously be $x \pm \propto$, and y; and consequently upon the same principle as in (XI.), the required formula for the new modulus is,

$$\text{Sin. T}' = \frac{y}{\overline{x \pm \propto}|^2 + y^2}\,. \qquad \text{(XII.)}$$

If, for example, the curve last considered had been commenced 3 chains *back* upon the tangent line AX, as at the point B, then the co-ordinates would have been $x + \propto = 25$, and $y = 12$; and, therefore,

$$\text{Sin. T}' = \frac{12}{625 + 144} = .01561\,;\ \text{or, T}' = 0^\circ\, 53\frac{3}{4}'\,;$$

which is the modulus required to trace a curve from the new origin at B, through the point F.

The direction of the tangent will now be considered. It is very evident that the two curves considered in the two last examples will have very different directions in passing the point F; and it is a matter of considerable importance in the field, after the rectangular lines a, b, c, &c., have been traced to any proposed point, to be able to examine, by the direction of the instrument, what the direction of the curve would be passing from the origin through that given point. Indeed, in different situations, a curve cannot be selected without such a datum ; and if the rectangular lines, a, b, c, &c., were not sufficient to furnish that datum with facility, a curve would have to be actually laid upon the ground in order to judge of its fitness, even if we knew a point F through which it would pass. A formula for this purpose is, however, easily obtained, from the rectangular lines a, b, c, &c. Let FI represent a tangent from the point F ; there would then, evidently, be no difficulty in directing the instrument, when placed at F, into the position FI, if we knew the inclination of that tangent to the original tangent AX, at the origin. For supposing KF to be the last rectangular line traced, it will, of course, be either parallel to AX or perpendicular to it ; and in *either case* it furnishes the means of directing the instrument into the line FH, parallel to the original tangent at the origin.—We have then only to deflect the angle HFI, equal to the inclination of the two tangents, when that inclination is known, and the direction of the curve at F may then be seen at once, from the position of the instrument, without that delay which would be occasioned by actually tracing a curve upon the ground, which would ultimately have to be relaid.

The result therefore is, that a formula must be investigated, expressing the inclination of the two tangents, in terms of the given co-ordinates x, y. Take D to denote the inclination required ; then by (IV.), $\mathrm{D} = 2n\mathrm{T}$ and therefore by (VII.),

$$x = \frac{\text{Sin. D}}{2 \text{ Sin. T}}, \text{ and, } y = \frac{1 - \text{Cos. D}}{2 \text{ Sin. T.}}$$

$$\text{Hence } \frac{\text{Sin. D}}{x} = \frac{1 - \text{Cos. D}}{y}; \text{ or } \frac{\text{Sin. D}}{1 - \text{Cos. D}} = \frac{x}{y}.$$

Now, substituting for $\frac{\text{Sin, D}}{1 - \text{Cos. D}}$, its value Cot. $\frac{1}{2}$D, agreeably to the principles of trigonometry, the following formula is the result,

$$\text{Cot. } \tfrac{1}{2}\text{D} = \frac{x}{y} \qquad \text{(XIII.)}$$

Example 1. Take the same curve which was proposed as an example for (XI.) In this case, then, the co-ordinates are $x =$ 22 chains, and $y =$ 12 chains; and therefore, Cot. $\frac{1}{2}$D $= \frac{22}{12} = 1.83333$; or by the table of natural cotangents, $\frac{1}{2}$D $=$ 28° 36$\frac{1}{2}$′; or D $=$ 57° 13′. Hence deflect the angle HFI $=$ 57° 13′, and the instrument will then indicate the true direction which the proposed curve would have in passing the point F, if traced from the origin at A, by means of the modulus of curvature before determined.

Example 2. Let the direction of the curve which was proposed as an example for (XII.) be required.

Here the co-ordinates are $x + \propto = 25$, and $y = 12$; and, therefore, Cot. $\frac{1}{2}$D $= \frac{25}{12} = 2.08333$. Hence, $\frac{1}{2}$D $=$ 25° 38$\frac{1}{2}$′, or, D $=$ 51° 17′; and consequently, in this case, deflect the angle HFI $=$ 51° 17′, to obtain the direction required.—This curve would therefore intersect the former, at the point F, with an angle of 5° 56′.

In all cases where both the angles D and T have to be found, the most convenient method will be to determine the value of D by means of (XIII.), and then compute the value of T from that of D. For by a reference to (IV.) and (VII.), the following theorem will be easily deduced,

$$\text{Sin. T} = \frac{\text{Sin. D}}{2\,x}. \qquad \text{(XIV.)}$$

17. It is frequently necessary that several points should be designated, through which a curve is required to pass, by means of a change of curvature at each of those points. To illustrate this case, and to show the method of operation which ought to be pursued under such circumstances, take AX, (see figure last

art.,) for the primitive tangent line, and AX, AY, for primitive co-ordinate axes, whose origin is the commencement of the required curve. Trace, parallel to those axes a system of rectangular lines given by the equations

$$\left\{\begin{array}{l} x = a + b + c + \&c. \\ y = d + e + f + \&c. \end{array}\right\},$$

and terminating at the *first designated point.* Let the instrument be then placed at that point, and directed into tangent agreeably to the method explained very fully in the last article. Take this second tangent as the axis of x, for a new system o rectangular co-ordinate axes ; and parallel to these new axes trace a second system of rectangular lines, given by the equations

$$\left\{\begin{array}{l} x = a' + b' + c' + \&c. \\ y = d' + e' + f' + \&c. \end{array}\right\},$$

and terminating at the *second designated point.* Let the instrument be now placed at this second point, and again directed into the proper tangent, by the same means as before. Take this third tangent as the axis of x, for a third system of rectangular co-ordinate axes ; and parallel to this second new system of axes, trace a third system of rectangular lines given by the equations

$$\left\{\begin{array}{l} x = a'' + b'' + c'' + \&c. \\ y = d'' + e'' + f'' + \&c. \end{array}\right\},$$

and terminating at the *third designated point.* Continue this obvious order of proceeding until all equations

$$\left\{\begin{array}{l} x = \qquad\qquad \\ y = \qquad\qquad \end{array}\right\}$$

have been obtained for *all the designated points* ; and then by means of those equations, and (XIV.), compute all the moduli of curvatures. Returning now with the instrument to the primitive origin at A, let each curve be traced from its proper modulus of curvature, agreeably to the principles explained in Art. 9 ; and the line will be found to pass through all the designated points.

If proper care be observed in chaining the different systems of rectangular lines by means of which the equations

$$\left\{\begin{array}{l} x = \qquad\qquad \\ y = \qquad\qquad \end{array}\right\}$$

have been obtained, there can be no disappointment in the result; and consequently, if the designated points have been judiciously selected, there will very seldom be a necessity of tracing the same part of a line *the second time.* And thus the method of co-ordinate axes, in the hands of an individual to whom that term is familiar, is susceptible of being made one of the most important facilities in the field, as will be further illustrated in subsequent articles.

In tracing the various systems of rectangular lines through the different points which may be designated for a curve, there is a principle of practical convenience which must be here mentioned. I mean the principle of designating such points, for a change of curvature, as will cause each section of the whole curve, between the designated points, to be composed of an *integer number of chains*, when those curves come to be ultimately traced, after their respective moduli of curvatures have been ascertained by the methods which have been already explained. It is indeed necessary, in every case, except where the roadway is perfectly horizontal, to know the length of each of those separate curves, in order to select the designated point judiciously with respect to the *grade;* and this datum must therefore always accompany the levels. When a system of those rectangular lines have been traced to any given point, and the corresponding equations

$$\left\{\begin{array}{l} x = \qquad\qquad \\ y = \qquad\qquad \end{array}\right\}$$

have been thus obtained, the distance from the origin to that given point, in a right line, will obviously be truly expressed by $\sqrt{x^2+y^2}$; which is a formula rendered very convenient for use, by means of the table of squares and square roots of numbers, subjoined to this volume. And this quantity may be frequently taken as the length of the intervening curve, by which to compute what the *grade* would be at that given point, and will always furnish an easy method of obtaining the approximate distance necessary in making a proper selection for the *position* o a line as far as the *levels have an influence.* The next objec then must be, finally, to designate such a point as near th point fixed by the levels as a desirable curvature will permit, an which will produce a curve, from the origin, containing an *integer* number of chains ; but as this last condition is only for convenience in subsequently tracing the curve, it must be don without injury to the line, which is in fact always practicable.

Suppose the first system of rectangular lines to be *terminate* at the first point selected from the nature of the ground. Let

$$\left\{\begin{array}{l} X = a + b + c \\ Y = d + e + f \end{array}\right\}$$

be the resulting equations, and compute the value of T, and th value of a certain angle D″, from the following formulas,

$$\text{Cot. } \tfrac{1}{2}\, D'' = \frac{X}{Y}$$

$$\text{Sin. } T = \frac{\text{Sin. } D''}{2X} \qquad \text{(XV.)}$$

Having now obtained the values of D″ and T, let a certai quantity, N, be found from the following expression,

$$N = \frac{D''}{2T}\,. \qquad \text{(XVI.)}$$

If, then, N be an integer number, it will express the numbe of chains in a curve passing from the origin to the point designated by the equations

$$\left\{\begin{array}{l} X = a + b + c \\ Y = d + e + f \end{array}\right\},$$

and the angle D″ will then give the inclination of the new tangen at that point, agreeably to (XIII.) : and T will express the modulus of curvature as appears from (XIV.). But if N is not a integer number, take *n*, the *nearest integer number to it*, and retaining the value of T, compute the corresponding *new co-ordinates*, x, y, by means of (VII.)

Finally, let two other rectangular lines $h = x - X$, and $k =$ $y - Y$, be traced from the point producing the equations

$$\left\{\begin{array}{l} X = a + b + c \\ Y = d + e + f \end{array}\right\},$$

and a new point

$$\left\{\begin{array}{l} x = a + b + c + h \\ y = d + e + f + k \end{array}\right\}$$

will be thus obtained, *to which* a curve being traced from the origin, by means of the modulus of curvature T, it will contain the integer number of chains denoted by *n*.

In a long curve it cannot be expected, in consequence of little inaccuracies which are unavoidable in chaining, deflecting, &c., that the stations will be found, when they come to be actually traced, to agree *exactly* with the point obtained for an integer chain, by the system of co-ordinate axes; but the difference will not be such as to produce any sensible derangement in subsequent parts of the curve. And it may be observed that the point

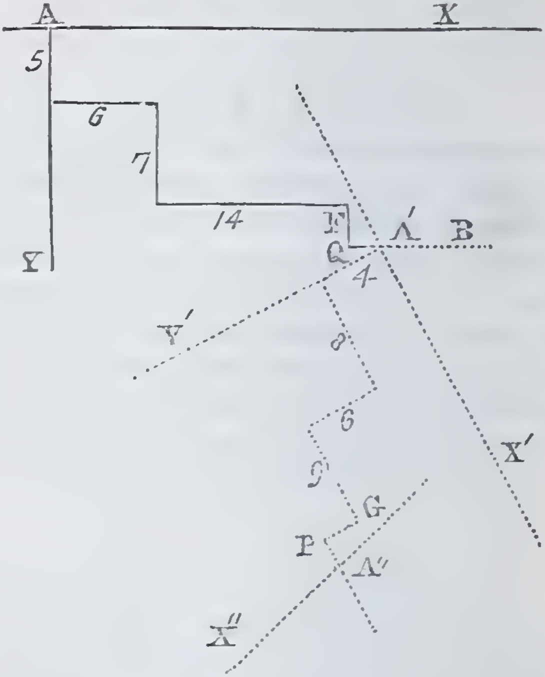
$$\left\{\begin{array}{l} x = a + b + c + h \\ y = d + e + f + h \end{array}\right\},$$

which was selected only for the sake of convenience in having the change of curvature to occur at the extremity of a *full chain*, when the whole line comes to be ultimately traced with the proper curvatures, will be situated within the nearest full chain of the given point

$$\left\{\begin{array}{l} X = a + b + c \\ Y = d + e + f \end{array}\right\};$$

and the same curve which meets the former point will also *pass through* the latter, if *n* be the nearest integer number *greater* than N.

The following example will more fully explain the principles contained in the present article. Let AX be a given tangent line, and A the given origin of a certain curve which is required to be laid in such a manner as to pass through the two points F and G, whose positions are given.



Take the lines indicated by the figures 5, 6, 7, and 14, to represent a system of rectangular lines, traced from the primitive origin at A, to the *first designated point* F ; and let those figures indicate the length of each line respectively in chains, as determined by measurement in the field. The equations for the point F will then be,

$$\left\{ \begin{array}{l} X = 6 + 14 \\ Y = 5 + 7 \end{array} \right\} ;$$

that is, X = 20 and Y = 12 ; and therefore by (XV.)

$$\text{Cot. } \tfrac{1}{2}D'' = \frac{20}{12} = 1.66667 ;$$

or, $\tfrac{1}{2}D'' = 30^\circ\ 58'$; or, $D'' = 61^\circ\ 56'$. And again by (XV.)

$$\text{Sin. } T = \frac{\text{Sin. } 61^\circ\ 56'}{40} = \frac{.88240}{40} = .02206 ;$$

or, $T = 1^\circ\ 16'$, which is the modulus of curvature necessary to trace a curve from the origin at A, passing through the first given point F. And now to determine whether the point F will coincide with an integer number of chains, in the curve, from the origin A, we have by (XVI.),

$$N = \frac{61^\circ\ 56'}{2^\circ\ 32'} = \frac{61.93}{2.53} = 24.48\,;$$

which is not an integer number. The point F, therefore, does not correspond to an integer number of chains in the curve; and a new point A′, must in consequence be found, which will coincide with the extremity of the 25th chain *in the same curve.* We then have $n = 25$, $T = 1^\circ\ 16'$; or $2\,n\,T = 63^\circ\ 20'$; and by (VII.)

$$x = \frac{\text{Sin. } 63^\circ\ 20'}{2 \text{ Sin. } 1^\circ\ 16'} = \frac{.89363}{.04422} = 20.21 \text{ chains,}$$

$$y = \frac{1 - \text{Cos. } 63^\circ\ 20'}{2 \text{ Sin. } 1^\circ\ 16'} = \frac{.55120}{.04422} = 12.47 \text{ chains.}$$

Hence $h = x - X = +\,0.21$, and $k = y - Y = +\,0.47$. Let, therefore, the two rectangular lines FQ and QA be traced in a manner similar to the former, and equal to .47 and .21 of a chain respectively, taking care to preserve their directions by references more remote; and a new station A′ will be thus obtained, coinciding with the extremity of the 25th chain in the same curve, which passes from the primitive origin at A through the first given point F. The instrument may now be placed at A′, and directed into the line A′B, parallel to the primitive tangent AX; and because $2\,n\,T = 73^\circ\ 20'$, therefore, agreeably to (IV.), deflect the angle $BA'X' = 63^\circ\ 20'$, and the instrument will then indicate the true direction A′X′, which is *tangent at the station* A′, to the curve subsequently to be traced from the primitive origin at A.

Now, take A′X′ and A′Y′, for a new system of rectangular coordinate axes, agreeably to which let a second system of rectangular lines be traced from the new origin at A′, and terminating in the *second given point* G, and indicated in chains by the figures

4, 8, 6, 9, respectively, as before. The equations for the point G will then be

$$\left\{ \begin{array}{l} X = 8 + 9 \\ Y = 4 + 6 \end{array} \right\};$$

that is, X = 17, and Y = 10; and therefore by (XV.)

$$\text{Cot. } \tfrac{1}{2}D'' = \frac{17}{10} = 1.70000;$$

or, $\tfrac{1}{2}D'' = 30° 28'$; or, $D'' = 60° 56'$. And again by (XV.),

$$\text{Sin. } T = \frac{\text{Sin. } 60° 56'}{34} = \frac{.87406}{34} = .02571;$$

or T = 1° 28′, which is the modulus of curvature necessary to trace a curve from the new origin at A′, passing through the second given point G. And now to determine whether the point G will coincide with the integer number of chains in the curve, from the new origin at A′, we have by (XVI.),

$$N = \frac{60° 56'}{2° 56'} = \frac{60.93}{2.93} = 20.80;$$

which is not an integer number. The point G, therefore, does not correspond to an integer number of chains in the curve; and a new point, A″, must in consequence be found, which will coincide with the extremity of the 21st chain, *in the same curve.*—We then have, $n = 21$, $T = 1° 28'$: or, $2\,n\,T = 61° 36'$; and by (VII.),

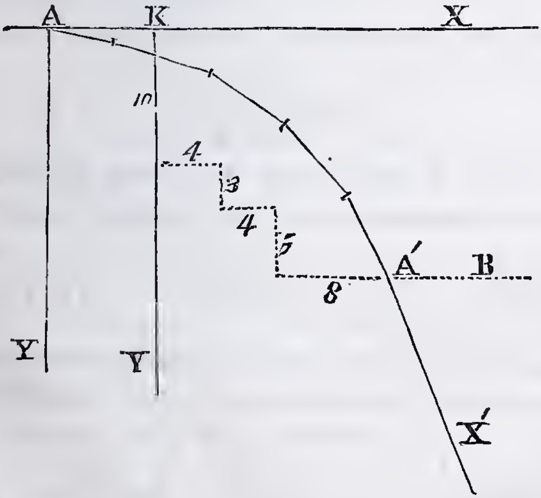
$$x = \frac{\text{Sin. } 61° 36'}{2 \text{ Sin. } 1° 28'} = \frac{.87965}{.05120} = 17.18 \text{ chains,}$$

$$y = \frac{1 - \text{Cos. } 61° 36'}{2 \text{ Sin. } 1° 36'} = \frac{.52438}{.05120} = 10.26 \text{ chains.}$$

Hence, $h = x - Y = +\,0.18$, and $k = y - Y = +\,0.24$. Let, therefore, the two rectangular lines GP and PA″ be traced in a manner similar to the former, and equal to .24 and .18 of a chain respectively; and a new station, A″, will be thus obtained, coinciding with the extremity of the 21st chain in the same curve which passes from the new origin A′, through the second given point G. The instrument may now be placed at A″, and directed into the line $A''D$, parallel to the first new tangent $A'X'$; and because $2\,n\,T = 61^\circ\,36'$, therefore, agreeably to (IV.), deflect the angle $DA''X'' = 61^\circ\,36'$, and the instrument will then indicate the true direction $A''X''$, which is *tangent at the station* A'', to the curve subsequently to be traced from its proper origin at A'.

The calculations of the two cases in this example have been *made out in full*, and expressed verbatim, with the exception only of such words as were peculiar to each case, in order that the principle might be at once comprehended by *all;* for it constitutes one of the most important *systems of operations* connected with the subject of curvatures, when properly understood and skilfully managed.

18. It is proposed to determine the modulus of curvature which will trace a curve from a given tangent line, and from a given *point therein*, in such a manner as to touch another right line given in position.



Let AX represent that tangent line in which the origin of the required curve is *unknown*, and $A'\,X'$ that in which the origin of the curve is given at the point A'. Select any arbitrary point K, in the *former* tangent line AX, for the origin of a system of rectangular co-ordinate axes KX, KY ; and parrallel respectively to those axes let a system of rectangular lines be traced from the origin of the co-ordinates at K, and terminating at that *origin of the curve* which is given in the other tangent line A′X′ at the point A′; and by means of those rectangular lines let the angle BA′X′ be *measured*, and denoted by D, and which will express the inclination of the two given tangents AX and A′X′ to each other. Take the equations

$$\left\{ \begin{array}{l} X = a + b + c + \&c. \\ Y = d + e + f + \&c. \end{array} \right\},$$

as *given* by means of those rectangular lines and explained in the preceding articles. It is then very evident that the value of the ordinate Y, as thus determined, will not, in any manner whatever, depend upon the point which was selected in the tangent line AX, for the *origin of the co-ordinates*; and consequently that value of Y will express the true ordinate which corresponds with the given point A′ in the required curve, as estimated from that *origin of the curve*, which is required to be situated somewhere in the given tangent line AX. And it hence follows from (VII.), that,

$$Y = \frac{1 - \text{Cos. } 2\,n\,T}{2 \text{ Sin. } T}.$$

But, D = 2 n T, as appears from (IV.) ; and this value being substituted in the expression for Y, the following formula will result :

$$\text{Sin. } T = \frac{1 - \text{Cos. } D}{2\,Y}. \qquad \text{(XVII.)}$$

And thus the modulus of curvature of the required curve may be easily computed after tracing a system of rectangular lines

$$\left\{ \begin{array}{l} X = a + b + c + \&c. \\ Y = d + e + f + \&c. \end{array} \right\}.$$

But that origin of the required curve, which is situated somewhere in the given tangent line AX, will also in some cases be wanted. Let that origin be at A, and put $\propto$, = AK ; and then X = $\propto$, and Y will evidently be the co-ordinates of the point A′

as estimated from the origin A, and axes AX, AY. Hence, by (XIII.)

$$\text{Cot. } \tfrac{1}{2}\text{D} = \frac{\text{X} = \propto}{\text{Y}};$$

or, eliminating $\propto$; the result is,

$$\propto = \text{Y} \times \text{Cot. } \tfrac{1}{2}\text{D.} - \text{X.} \qquad \text{(XVIII.)}$$

Which makes the required origin known; but in using this formula, it must be observed that the hypothesis here assumed is, that $\propto$ is considered *positive* when it is measured immediatdly *opposite* to the direction of X from the selected origin of the given co-ordinates X and Y; and consequently, when $\propto$ comes out *negative* from (XVIII.), it must be taken in a direction *coinciding* with X.

Example. Let a system of rectangular lines be traced in a manner similar to those represented upon the figure by the figures 10, 4, 3, &c.; and let those figures denote the length of each line in chains. The angle BA′X′ being measured, let it be found = 60°

In this case, then,

$$\left\{ \begin{array}{l} \text{X} \quad 4 = +4 + 8 \\ \text{Y} = 10 + 3 + 5 \end{array} \right\};$$

that is, X = 16, and Y = 18. And,

$$\text{Sin. T.} = \frac{1 - \text{Cos. } 60^\circ}{36} = \frac{.5000}{36} = .01390;$$

from which is found $\text{T} = 0^\circ\ 48' =$ modulus of curvature of the required curve. To find the position of the origin A, we have $\propto = 18 \times \text{Cot. } 30^\circ - 16 = 18 \times 1.732 - 16 = 31.18 - 16 = +15.18$. Hence measures 15.18 chains from the selected point K, and *backwards* as regards the axis KX, of X. The required point A, will be then obtained; from which a curve being traced by means of the modulus of curvature 0° 48′, it will meet the given point A′, and come into the given tangent line A′X′. Or, vice versa, being traced from the given point A′ and from the given tangent line A′X′, it will *touch* the other given right line AX, at the point A.

It frequently happens, in tracing curves for the superstructure of a railroad upon a graded surface, that AX will represent a *fixed* tangent, and that the point A′ will represent a station in a curve which is also *fixed*, and to which A′X′ is a tangent at that point ; and it is then required to pass that curve into the curve AA′ at the station A′ and upon the *common* tangent A′X′. In such a case it will always be the best method, after the necessary modulus of curvature has been computed by means of (XVII.), to trace the curve from A′ towards the other origin at A. For then the two curves will meet at A′ with an integer chain ; and the broken chain will come in more conveniently, at the other extremity of the curve, where it meets the given tangent line AX.

The formulas (XVII.) and (XVIII.) will also, in some cases be highly useful in conducting the location of a line with facility. Suppose, for example, that AX be a given tangent line, whose direction is fixed by some former curve, and particular local circumstances ; and let the line A′X′ represent the direction of some bluff, or other obstacle, which makes it desirable for that part of the contemplated road to occupy ground which will correspond with the line A′X′, or differ from it as little as circumstances will permit. In this case, then, having selected any convenient point K, in the given tangent line AX, for the origin of co-ordinate axes, let the system of rectangular lines,

$$\left\{ \begin{array}{l} X = a + b + c + \&c. \\ Y = d + e + f + \&c. \end{array} \right\},$$

be traced from that origin, and terminating in such a point A′ as the nature of the ground *and the levels* may require. Then placing the instrument at A′, deflect into the proper direction A′X′, and thus measure the angle BA′X′. Having then all the requisite data for computing the origin A, and the necessary modulus of curvature, to pass a curve through the point A′ and into the direction A′X′, let those items be calculated. It will then be seen at once, whether the point A′, and the direction A′X′, can both be maintained within the limits of the desired expense and curvatures ; and if not, then either some other point

A′ must be taken, or some new direction must be assumed.— These facts should be always ascertained, and the most judicious line definitely selected, *before any curve is actually traced.* It is, however, necessary to observe here, that, in the location of a line preparatory to the graduation, it will be very seldom proper to trace a curve backward from any selected point A′ to meet the line upon a given tangent AX, as was recommended in the case of a graded surface. For in the first location, integer chains should be always retained when practicable ; and which cannot be easily done without tracing the curves and tangent lines continuously. In the case, then, last considered, the following principles should be observed : After the point A′ and the direction A′X′ have been finally selected, and the requisite origin A, together with the modulus of curvature for the required curve, have been already calculated, if that origin does not fall upon the tangent line AX at an integer chain, then select the nearest integer chain for the true origin ; and compute a *new modulus of curvature*, which will pass a curve from that latter origin through the designated point A′. It is then very evident that the direction of this curve, in passing the point A′, will differ but little from the direction of the former curve at the same point. When therefore the curve, as last selected, has been laid within one ortwo chains of the designated point A′, let such a variation be made in the modulus of curvature, for the remaining chain or two, as will restore the *parellelism* of the two tangents, agreeably to principles which will be obvious enough from (V.) In this manner the true direction will be obtained ; and although the line will not then pass exactly through the designated point A′, yet it will not vary a material distance from it.

When this system of operation is to be pursued, it will always be convenient, in selecting the arbitrary point K, to take it at an interger chain, counting from the termination of the line as actually laid into the given tangent AX ; and then if α does not appear from (XVIII.) to be an integer number of chains, select the nearest integer number to it, and the co-ordinates by which to compute the *new modulus of curvature*, from (XII.) will evidently be $X + \propto$, and Y, regard being had to the proper sign of $\propto$, as already explained.

19. Although a system of rectangular lines, traced from given co-ordinate axes, will in general furnish the best data for computation, yet cases sometimes occur when those calculations have to be made from computed curves or curves already laid upon the ground. In a first location, this case will sometimes happen, when, from difficulties which are found in advance of a line, it becomes necessary to change a part of that which was either already computed, or actually laid. Such a case will sometimes occur even when the operations in the field have been skilfully conducted ; and in laying curves upon a surface already graded, it will be frequently necessary to compute from curves actually traced ; for in that case it will be often required to notice the particular situations of several stations in a curve or tangent line, and which cannot be so readily done in any other manner as by actually laying an approximate line upon the ground. The *notes* of those approximate lines are, moreover, in such cases, *always furnished* from a former survey.

20. Suppose two curves to be laid upon the same tangent line, and let $\propto$ denote the distance between their origins. Let one of those curves have a given modulus of curvature denoted by T, and let it pass through a certain given point at the extremity of the nth chain. It is then proposed to find the modulus of curvature of the other curve, such that it may also pass through the same given point; and it is likewise required to determine the angle of intersection of the two curves at that point.

The co-ordinates of the given point, taken with reference to the origin of the required curve, are obviously $x \pm \propto$ and y ; and consequently, taking T′ to denote the required modulus of curvature, its value will be expressed by

$$\text{Sin. } T' = \frac{y}{x^2 + y^2 \pm 2 \propto x + \propto^2},$$

as appears from (XII.). Hence, substituting in this expression for x and y, their values* as obtained by (VII.), the following theorem will be the obvious result :

$$\text{Sin. T}' = \frac{1 - \text{Cos. } 2n\text{T}}{\frac{1 \text{ Cos. } 2n\text{T}}{\text{Sin. T}} \pm 2 \propto. \text{ Sin. } 2n\text{T} + 2 \propto^2. \text{ Sin. T}}. \qquad \text{(XIX.)}$$

The formula thus obtained will be found to be quite convenient for purposes in the field, when the curves are too long to be within the limits of the more simple approximating methods, which will be hereafter explained.

With regard to the double sign, it must be observed that the *negative value of* ∝ when the origin of the curve sought is in *advance* of the origin of the given curve, with respect to the direction in which the curves are laid from their origins ; and the *positive* value must be taken in the contrary case.

In order to determine the angle of intersection of the two curves, take D′ to denote the number of degrees and minutes contained in the curve whose modulus of curvature is T′, between its origin and the given point, and which will consequently, by (I.) and (IV.) also express the inclination of the required curve, at that point, to the *common tangent* at the origins. The value of D′ will then be expressed by

$$\text{Cot. } \tfrac{1}{2}\text{D} = \frac{x \pm \propto}{y},$$

as appears from (XIII.). Now, substituting in this expression for x and y, their values as given by (VII.), the following formula will be the result :

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* NOTE.—By squaring each of the two formulas given by (VII.), and adding the two squares together, the following expression will result after a little reduction :

$$x^2 + y^2 = \frac{1 - \text{Cos. } 2n\text{T}}{2 \text{ Sin. }^2\text{T}}$$

This is, however, not the form which a formula should have for computing the numerical value of $x^2 + y^2$; but it is useful in reducing other formulas.

$$\text{Cot. } \tfrac{1}{2}D' = \text{Cot. } nT \pm 2 \propto . \frac{\text{Sin. } T}{1 - \text{Cos. } 2nT}. \qquad \text{(XX.)}$$

The value of D′ being thus obtained, the required angle of intersection will be expressed by $2nT - D'$; as evidently appears from (V.)

Example 1. Let the modulus of curvature of the given curve be 30′, and let the given point be at the extremity of the 60th chain. Supposing the origin of the other curve to be placed 10 chains *back* upon the tangent line, what will be the new modulus of curvature, and what will be the angle of intersection at the given point?

In this case,

$$\text{Sin. } T' = \frac{.50000}{\frac{.50000}{.00873} + 20 \times .86603 + 200 \times .00873}$$

$$= \frac{.50000}{57.299 + 17.321 + 1.746} = \frac{.50000}{76.366} = .00655\,;$$

and consequently, $T' = 0° \; 22\tfrac{1}{2}'$ = required modulus of curvature. And, $\text{Cot. } \tfrac{1}{2}D' = 1.73205 + 20 \times \frac{.00873}{.50000} = 1.73205 + 20 \times .01746 = 1.73205 + .34020 = 2.08125$; and therefore, $\tfrac{1}{2}D' = 25° \; 40'$; or, $D' = 51° \; 20'$. Hence, $2nT - D' = 60° - 51° \; 20' = 8° \; 40' =$, the angle of intersection required.

Example 2. Let the modulus of curvature of the given curve be 2°, and let the given point be at the extremity of the 31st chain. Suppose the new curve to be commenced one chain in *advance* upon the tangent line, what will be the required modulus of curvature, and what will be the angle of intersection at the given point?

Here, $T = 2°$, and $n = 31$; and consequently $2nT = 124°$. Hence, in this case, Cos. $2nT$ becomes *negative*, while Sin. $2nT$ remains positive, agreeably to the principles of trigonometry. Also, in the present example, $\propto = -1$. Under those considerations, the formula (XIX.) gives,

$$\text{Sin. } T' = \frac{1 + \text{Cos. } 56^\circ}{\dfrac{1 + \text{Cos. } 56^\circ}{\text{Sin. } 2^\circ} - 2 \text{ Sin. } 56^\circ + 2 \text{ Sin. } 2^\circ}$$

$$= \frac{1.55919}{\dfrac{1.55919}{.03490} - 2 \times .82904 + 2 \times .03490}$$

$$= \frac{1.55919}{44.676 - 1.685 + .070} = \frac{1.55919}{42.788} = .03643\,;$$

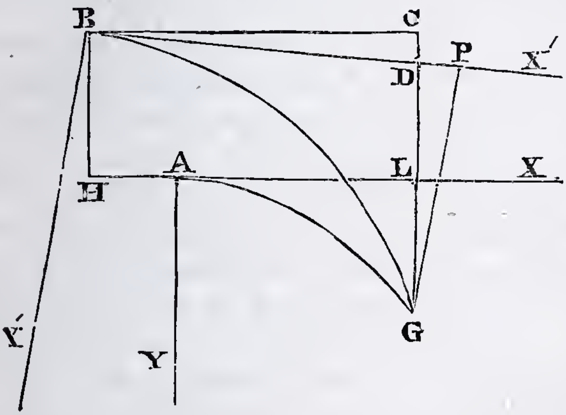
or, $T' = 2^\circ\ 5\frac{1}{4}'$ = required modulus of curvature. In obtainng the value of the angle D' by means of (XX.), the necessary sign of the trigonometrical quantities must in like manner be observed; and consequently, $\text{Cot. } \frac{1}{2}D' = \text{Cot. } 62^\circ - 2 \times \dfrac{\text{Sin. } 2^\circ}{1 + \text{Cos. } 56^\circ} = .53171 - 2 \times \dfrac{.03490}{1.55919} = .53171 - 2 \times .02239 = .53171 - 04478 = .48693$ or $\frac{1}{2}D' = 64^\circ\ 2'$; and $D' = 128^\circ\ 4'$. Hence, $D' - 2nT = 4^\circ\ 4'$, the angle of intersection required.

In computing the modulus of curvature for a long curve, which is required to be laid with precision, it will not be sufficient to obtain a result only true to the nearest *full minute.* For in tracing a curve agreeably to the principles given in Art. 3, it is evident that multiplies of the modulus of curvature are required in laying the different chains; and consequently, an error of $\frac{1}{4}'$ in the modulus of curvature would produce an error of 2′ in the angle of deflection, placing the 8th chain from the instrument. Now, in any common instrument, containing a good vernier scale, an angle of 2′ is quite perceptible. Indeed, if the chains be 100 feet each, an angle of 2′ will be subtended by a line nearly six inches long at a distance of eight chains; and it follows, therefore, that an error of *one fourth part of a minute*, in the modulus of curvature, will, in such a case, produce an error of nearly six inches in the position of every 8th station from the instrument. And as the effects of this error will not continue to be only proportionably greater through the whole line, but will accumulate faster and faster, in a geometrical ratio, it is evident that an error of $\frac{1}{4}'$ in the modulus of curvature is an item of considerable consequence in laying long curves upon a graded surface.

21. If the point designated for the required curve to meet does not coincide with the extremity of the nth chain in the given curve, as in the last article it is supposed, but varies a small distance to the right or left, yet, *if the curves are long*, the best method will always be to compute the value of T′ as though the curve sought were intended to pass through the point contemplated in (XIX.) ; and then the requisite small variation in the computed value of T′, to meet the case proposed, may be subsequently determined by very simple methods, to be hereafter explained.

22. Let two curves be under consideration having different origins, and tangent lines ; and let one of those curves be given, and a point designated therein, through which the other curve is required to pass. Take a system of rectangular co-ordinate axes, corresponding with the given origin, and tangent line, of the given curve, and let the co-ordinates of that point which is designated for the required curve to meet be x, y; the value of these co-ordinates being computed by means of (VII.) if the given curve be already laid in the field, but determined by means of a system of rectangular lines when that curve has not been actually laid. Let the co-ordinates of the *new origin*, taken with reference to the axes of x, y, and determined either by computation or by means of a system of rectangular lines, be denoted by α β ; α *being supposed to coincide with the axis of* x. And lastly take z to denote the given inclination of the tangents at the origins of the two curves. From those data it is then proposed to find the modulus of curvature of the required curve, such that it may pass through the designated point.

Take a new system of rectangular co-ordinate axes, corresponding with the origin and tangent line of the required curve ; and agreeably to these, let x', y', represent the new co-ordinates of the point designated in the given curve. It is then very obious that the required modulus of curvature will be immediately erived from (XI.), when the new co-ordinates x', y', become nown. The value of those co-ordinates must therefore be ought agreeably to the known methods of analytical geometry, or the transformation of co-ordinates. Let A be the origin and G the designated point, in the given curve AG; and take the oint B for the new origin of the required curve BG, and AX, AY, and BX′, BY′, for the two given systems of rectangular co-rdinate axes; the two tangent lines coinciding with AX and BX′ respectively. Let such lines be drawn as appear obvious upon the figure, and the following values will then obtain $x =$ AL, $y =$ LG, $\propto =$ AH, $\beta =$ HB, $x' =$ BP, $y' =$ PG, and z $= \angle$ CBD $= \angle$ LGP.



By plane trigonometry CD $= \overline{x + \propto}\cdot$ Tan. z; and consequently DG $= y + \beta - \overline{x + \propto}\cdot$ Tan. z; but again by plain trigonometry, PG $=$ DG$\cdot$ Cos. z. Hence, PG $= \overline{y + \beta}\cdot$ Cos. $z - \overline{x + \propto}\cdot$ Sin. z. In like manner it will be found that BP $= \overline{y + \beta}\cdot$ Sin. $z + \overline{x + \propto}\cdot$ Cos. z. The values of the new co-ordinates will therefore be expressed as follows:

$$x' = \overline{y + \beta}\cdot \text{Sin.}\, z + \overline{x + \propto}\cdot \text{Cos.}\, z$$
$$y' = \overline{y + \beta}\cdot \text{Cos.}\, z - \overline{x + \propto}\cdot \text{Sin.}\, z. \qquad \text{(XXI.)}$$

Such are the formulas to be used in the field, when a new system of co-ordinates must be computed ; they are the well known expressions given by most authors for the transformation of rectangular co-ordinates, and they only here stand transposed in such a manner as will best suit the engineer's purpose in the present inquiry.

It may be observed, that the value of the angle z need never be obtained by a measurement with the instrument ; for it may always be easily computed from the manner in which the two origins, at A and B, have been obtained.

The value of the new co-ordinates having been found, the required modulus of curvature may be easily computed by means of (XI.) as before remarked. But a direct formula will be more convenient for use ; and in order to obtain such a formula, let each of (XXI.) be squared, and the result will give $x'^2 + y'^2 = \overline{y + \beta}\,|^2 + \overline{x + \propto}\,|^2$. Hence denoting the required modulus of curvature by T′, the following formula will be immediately derived from (XI.)

$$\text{Sin. T}' = \frac{\overline{y + \beta}\cdot \text{Cos. } z - \overline{x + \propto}\ \text{Sin. } z}{\overline{y + \beta}\,|^2 + \overline{x + \propto}\,|^2}. \qquad \text{(XXII.)}$$

The theorem thus obtained, expressing the value of T′, has a very good form for numerical computations, and when skilfully applied, it will frequently save much labor in the field, which would be otherwise required when *certain alterations* are proposed in a line *once computed, or actually traced.* And although the co-ordinates $\propto$, β, and the angle z, which expresses the inclination of the axis of x, and of x', to each other, will change their *signs* under different circumstances in the field, yet, to those who are familiar with the use of algebraic formulas, this cannot be a source of any embarrassment. For it has only to be observed that the hypothesis here assumed is, that $\propto$ is accounted *positive* when its direction is immediately *opposite* to the direction of x ; and β is, in like manner, accounted *positive* when its direction is immediately *opposite* to the direction of y. And either $\propto$ or β must, in consequence, have its sign *reversed* in (XXI.) or (XXII.), when the circumstances in the field are such as to give either of them the same direction with its respective co-ordinate axis.

The angle z is to be accounted negative, or which is the same thing, the quantity Sin. z must have its sign *reversed*,* in the two following cases: I. When β is positive, and the two tangents diverge *in advance.* II. When β is negative, and the two tangents converge *in advance.* In all other cases the formulas (XXI.) and (XXII.) will retain their present forms, as far as the angle z is concerned.

In order to show *one* case in which an application of (XXII.) will be extremely convenient in practice, let TRS be a curve already laid in the field *upon such ground as ought to be selected*, and let SA be a *short* tangent intervening between the given curve TRS, and a certain reserved curve AG, necessary to pass a designated point G. Having traced a system of rectangular lines from the given origin A, and terminating in the designated point G, let the modulus of curvature be computed by means of (XI.) which would trace the curve AG, and let the direction of that curve, at the point G, be examined agreeably to the method explained in Article 16. Now, supposing this curvature is found more abrupt than is thought to be judicious, the only method of alteration will be to take the curve TRS off into a tangent a little sooner, as for instance at the station R, and then a new origin will be obtained at B, in the new tangent RB, for the required curve BG, which is the very case under consideration in the present article. The inclination of the two tangents RB and SA, will be known at once from (IV.); and the co-ordinates of the new origin B may be easily computed by methods to be hereafter explained. By repeating a calculation from (XXII.) for several points in the given curve TRS, it will be easy to select a proper point R, at which to terminate the given curve TR, in order to lay a short tangent RB in such a position as to meet the necessary conditions imposed by the reversed curve BG. Other cases will be hereafter mentioned, in which an application of (XXII.) will be required; but, in the first place, the following examples are here given, amply to illustrate the *various mutations of that formula, under the different circumstances occurring in practice.*

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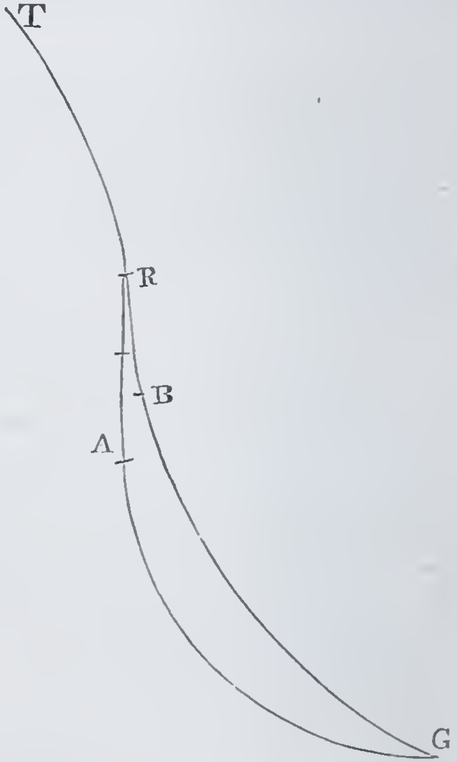
* Note.—Agreeably to the principles of trigonometry, when an arc becomes negative, the Sine becomes negative also; but the Cosine does not change its sign.



Example 1. Let figures 1 and 2 exhibit the relative positions of the origins and tangent lines in two *different instances* occurring in the field; A being the primitive origin in both figures. Suppose a system of rectangular lines to be traced from the origin A, and terminate in the designated point G, and let the resulting equations give, $x = 17$ chains, $y = 12$ chains, agreeably to

the principles explained in Art. 16. Take also the co-ordinates AH and HB, of the *new origin*, equal to 3 and 2 chains respec-ively, and let the inclination of the new tangent BX′ be 10°, *con-verging* in advance.

Fig. 1.

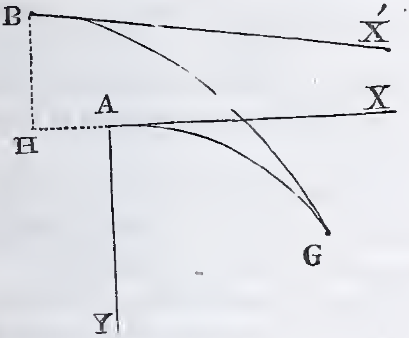
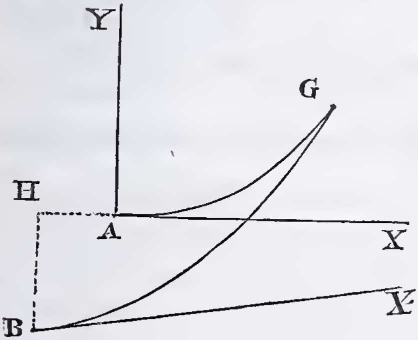


Fig. 2.



In this case, either figure being under consideration in the field, the following values will obtain, viz.: $\alpha = + 3$, $\beta = + 2$, and $z = + 10°$. Hence,

$$\text{Sin. T}' = \frac{\overline{12+2}\cdot\text{Cos. }10^\circ - \overline{17+3}\cdot\text{ Sin. }10^\circ}{\overline{12+2}|^2 + \overline{17+3}|^2}$$

$$= \frac{14\cdot\text{Cos. }10^\circ - 20\cdot\text{Sin. }10^\circ}{196+400}$$

$$= \frac{14\times\cdot 98481 - 20\times\cdot 17365}{596}$$

$$= \frac{13\cdot 787734 - 3\cdot 47300}{596} = \frac{10\cdot 31434}{596}$$

$= \cdot 01731$; or, $\text{T}' = 0^\circ\, 59\tfrac{1}{2}'$ = modulus of curvature necessary to trace a curve from the *new origin* B to the designated point G.

Example 2. Let the relative positions of the *origins* in two other instances be the same as in figures example 1 ; and suppose the two tangents to *diverge* in advance.

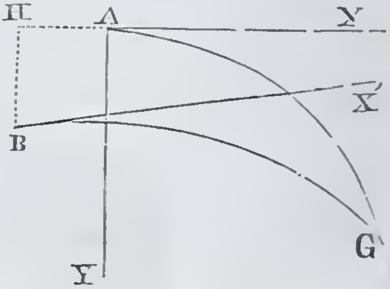
Here, retaining the same quantities given in the first example, the signs which appertain equally to fig. 1 and fig. 2 are the following viz. : $\propto\, = +3$, $\beta = +2$, and $z = -10^\circ$. And therefore, in this case,

$$\text{Sin. T}' = \frac{\overline{12+2}\cdot\text{Cos. }10^\circ + \overline{17+3}\cdot\text{Sin. }10^\circ}{\overline{12+2}|^2 + \overline{17+3}|^2}$$

$$= \frac{13\cdot 78734 + 3\cdot 47300}{596} = \frac{17\cdot 26034}{596} = \cdot 02896\,;$$

or, $\text{T}' = 1^\circ\, 39\tfrac{1}{2}'$ = modulus of curvature required.

Fig. 3.



Example 3. Let the relative positions of the origins and tangent lines be such as is indicated by figures 3 and 4, the primi-

tive origin being representnd at A, as before, and the tangents AX, BX′, *converging* in advance. It is evident that the same signs will appertain to both of the figures 3 and 4; that is $\propto = +3$, $B = -2$, and $z = -10°$. And therefore in this case,

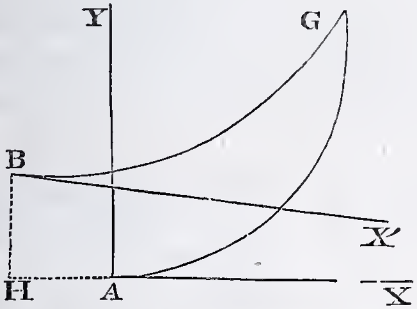
$$\text{Sin. } T' = \frac{\overline{12 - 2} \cdot \text{Cos. } 10^\circ + \overline{17 + 3} \cdot \text{Sin. } 10^\circ}{\overline{12 - 2}|^2 + \overline{17 + 3}|^2}$$

$$= \frac{10 \times \cdot 98481 + 20 \times \cdot 17365}{100 + 400} = \frac{9 \cdot 84810 + 3 \cdot 47300}{500}$$

$$= \frac{13 \cdot 32110}{500} = \cdot 02664\ ;$$

or, $T' = 1^\circ\ 31\tfrac{3}{4}'$ = modulus of curvature, which would trace ither of the curves, in figure 3 or figure 4, from the *new origin* B to the designated point G.

Fig. 4.



Example 4. Suppose the relative position of the *new origin* to be such as is represented by figure 3, or figure 4; and let the two tangents AX and BX′ *diverge* in advance.

Taking the signs which appertain to this case, the following values will obtain: $\propto = +3$, $B = -2$, and $z = +10°$; and consequently

$$\text{Sin. } T' = \frac{\overline{12 - 2} \cdot \text{Cos. } 10^\circ - \overline{17 + 3} \cdot \text{Sin. } 10^\circ}{\overline{12 - 2}|^2 + \overline{17 + 3}|^2}$$

$$= \frac{9 \cdot 84810 - 3 \cdot 47300}{500} = \frac{6 \cdot 37510}{500} = \cdot 01275\ ;$$

or, $T' = 0° \, 43\frac{3}{4}'$ = new modulus of curvature which would be required in either of the figures 3 or 4, upon the supposition tha the tangents AX and BX′ *diverge* in advance.

Fig. 5.

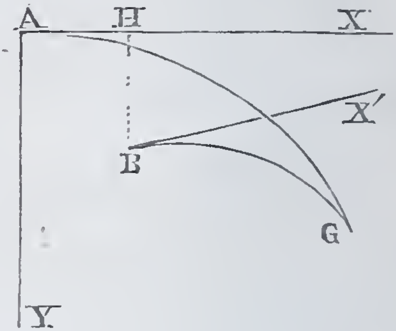
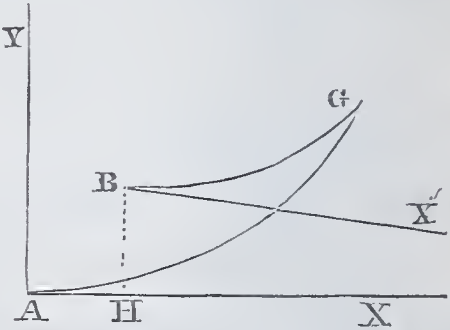


Fig. 6.



Example 5. Let figures 5 and 6 show the relative positions o the origins and tangent lines, as occurring in two different instances; the tangents being represented *converging* in advance.

Here the following values will obviously obtain with reference to either of the figures 5 or 6; that is, $\propto = -3$, $B = -2$ and $z = -10°$. Therefore,

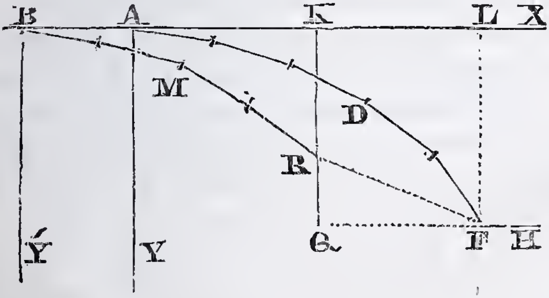
$$\text{Sin. } T' = \frac{\overline{12-2} \cdot \text{Cos. } 10° + \overline{17-3} \cdot \text{Sin. } 10°}{\overline{12-2}|^2 + \overline{17-3}|^2}$$

$$= \frac{9{\cdot}84810 + 2{\cdot}43110}{296} = \frac{12{\cdot}27920}{296} = {\cdot}04148;$$

or, $T' = 2° \; 22\frac{3}{4}'$ = required modulus of curvature.

The preceding examples do not exhibit all the mutations which will sometimes occur in (XXII)—but are sufficient to explain the principle. The importance of that formula in the field rendered such explanations necessary, in order that those persons who may not be already familiar with that elementary principle of analytical geometry, may nevertheless be enabled to compute without perplexity or liability to error.

23. Suppose ADF to represent a given curve, and BMR another proposed curve laid upon the same tangent line AX, and let $\propto$ denote the given distance AB, between their origins.—Take T and T′ to represent the given moduli of curvatures, and n, and m, the given number of chains contained in each curve respectively. It is required to determine the distance FR, between the extreme stations of those two curves; and it is also proposed to show a method by which the instrument may be directed into the line FR, from the extremity of the given curve ADF.



Take two systems of rectangular co-ordinate axes, AX, AY, and BX, BY′, corresponding with the common tangent line AX, and whose origins coincide respectively with the origins A and B of the two curves. Taking x, y, to denote the co-ordinates AL, LF, of the extremity of the given curve ADF, their values will be either known from (VII.), or determined in the field by means of a system of rectangular lines. Let x', y', be the co-ordinates, BK, KR, of the extremity of the proposed curve BMR, as estimated agreeably to the proper axes BX, BY′. The differences, KL, RQ, of co-ordinates, will then be expressed by $x + \propto - x'$, and $y - y'$, respectively. Hence, taking w, to denote the distance sought, the common principle of analytical geometry gives the following formula:

$$w = \left\{ \overline{x + \propto - x'}\big|^2 + \overline{y - y'}\big|^2 . \right\}^{\frac{1}{2}} \qquad \text{(XXIII.)}$$

And thus the distance w becomes known; for the values of the co-ordinates x', y', will be expressed as follows:

$$x' = \frac{\text{Sin. } 2\, m\, \text{T}'}{2 \text{ Sin. T}'}$$

$$y' = \frac{1 - \text{Cos. } 2\, m\, \text{T}'}{2 \text{ Sin. T}'}.$$

Now, in order to determine a method of directing the instrument into the line FR, when placed at the given station F, let FH be a line whose direction is parallel to the axis of x, and whose position is in *advance* of the station F, with reference to the origin A, upon the axis AX. The instrument may then be directed into the line FH, either by means of (IV.), or by the principles contained in Art. 16, agreeably to the circumstances of the case. Take, therefore, the letter P, to denote the angle of position HFR; and when its value has been computed, by a method which will be presently shown, deflect the angle HFR = P, and the instrument will indicate the required direction FR. And thus the position of the station at R will be seen at once, without, actually tracing the proposed curve BMR.

The value of the angle P is now to be investigated. By plane trig. Rad. : Sin. QFR : : FR : RQ; and, also, Rad. : Cos. QFR : : FR : FQ. That is,

$$\text{Sin. QFR} = \frac{y - y'}{w},$$

and

$$\text{Cos. QFR} = \frac{x + \propto - x'}{w}.$$

But, agreeably to the principles of trigonometry, Sin. P = Sin. QFR, and Cos. P = — Cos. QFR. Hence the following formulas are the obvious result:

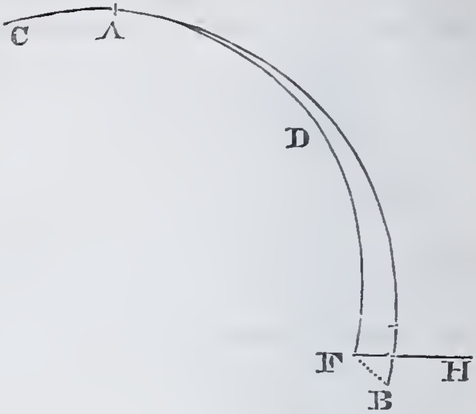
$$\text{Sin. P} = \frac{y - y'}{w};$$

$$\text{Cos. P} = \frac{x' - x - \propto}{w}. \qquad \text{(XXIV.)}$$

It will require a reference to both of the expressions (XXIV.), to determine the magnitude of the angle P, although one of them will be sufficient in finally making the computation. In this inquiry, the *signs* of the various quantities must be particularly observed in the following manner.

The quantity $\propto$ is considered positive when the given origin A is in *advance* of the proposed origin B; and consequently, *in a contrary case*, the quantity $\propto$ must have its sign *reversed* in (XXIII.) and (XXIV.). The angle P is supposed to be measured from the station F, and from the line FH, in a direction increasing from *right to left*, with reference to the line FH, through all four of the quadrants. By observing this principle, the position of the line FH, under all circumstances, will be immediately known from the *signs* of the quantities Sin. P and Cos. P. Thus, when Sin. P and Cos. P are both positive, the line FR will be situated in the first quadrant; when Sin. P is positive, and Cos. P negative, then is the line FR posited in the second quadrant; when Sin. P and Cos. P are both negative, then will the line FR fall in the third quadrant; and lastly, when Sin. P is negative, and Cos. P positive, then will the situation of the line FR be found in the fourth quadrant.

The following example will illustrate the utility of (XXIII.) and (XXIV.)



Example. Let CADF be a curve already traced from a modulus of curvature of 2° 30′, and let A be a station in that curve 30 chains from the extreme station at F. Supposing a change of curvature to be made at A, such as to trace the curve AR, for a distance of 30 chains, from a modulus of curvature of 2° 28′, it is proposed to know how far those two curves would be separated from each other at their extremities; and it is also required to designate a line immediately from the extremity of the given curve, as already laid, to the point where the proposed curve would terminate.

In this case, then, $\propto = o$, $n = m = 30$, $T = 2° 30'$, and $T' = 2° 28'$. Hence, $2\,n\,T = 150°$, $2\,m\,T = 148°$; and therefore, by (VII.),

$$x = \frac{\text{Sin. } 150°}{2 \text{ Sin. } 2° 30'} = \frac{\cdot 50000}{\cdot 08724} = 5\cdot 731\,;$$

$$y = \frac{1 - \text{Cos. } 150°}{2 \text{ Sin. } 2° 30'} = \frac{1 + \cdot 86603}{\cdot 08724} = \frac{1\cdot 86603}{\cdot 08723} = 21\cdot 389\,;$$

$$x' = \frac{\text{Sin. } 148°}{2 \text{ Sin. } 2° 28'} = \frac{\cdot 52992}{\cdot 08608} = 6\cdot 156\,;$$

$$y' = \frac{1 - \text{Cos. } 148°}{2 \text{ Sin. } 2° 28'} = \frac{1 + \cdot 84805}{\cdot 08608} = \frac{1\cdot 84805}{\cdot 08608} = 21\cdot 469.$$

Wherefore, $x + \propto - x' = 5{\cdot}731 - 6{\cdot}156 = -{\cdot}425$; $y - y' = 21{\cdot}389 - 21{\cdot}469 = -{\cdot}080$; and $w = \overline{\overline{-{\cdot}425}|^2 + \overline{-{\cdot}080}|^2}|^{\frac{1}{2}} = \overline{{\cdot}1806 + {\cdot}0064}|^{\frac{1}{2}} = \overline{{\cdot}1870}|^{\frac{1}{2}}$ $= {\cdot}433$ of a chain $=$ required distance FR. In this example, the position of the line FH, which is parallel to a tangent at A, must be obtained agreeably to the method explained in Art. 11; and then,

$$\text{Sin. P} = \frac{y - y'}{w} = \frac{-{\cdot}080}{{\cdot}433} = -{\cdot}1848;$$

$$\text{Cos. P.} = \frac{x' - x - \propto}{w} = \frac{+{\cdot}425}{w}.$$

Here, Sin. P is negative, and Cos. P positive; and consequently the angle P falls in the 4th quadrant; that is, from Sin. P $= -{\cdot}1848$, I find, P $= 349° 21'$, which is the value of the *exterior* angle HFR; and therefore deflect the *interior* angle HFR equal to $10° 39'$, and the instrument will then show the required direction of the line FR.

24. When the given curve ADF, (see fig. art. 23,) has been actually traced in the field, the co-ordinates x, y, have to be computed by means of (VII.) in order to obtain the distance FR, as proposed in the last article. In such a case, *if the two moduli of curvatures* T *and* T′ *be equal to each other*, the distance FR and the angle P will be more conveniently had by means of a direct formula in terms of n, m, T, and $\propto$, without first computing the values of the co-ordinates $x\ y$, and $x'\ y'$. For when (XXIII.) is developed, agreeably to the common principles of algebra, the result is, $w = (x^2 + y^2 + x'^2 + y'^2 - 2xx' - 2\,yy' - 2 \propto . \overline{x' - x} + \propto^2)^{\frac{1}{2}}$; and by substituting in this equation for x, y, x', y', their values obtained from (VII.), upon the supposition that $\mathrm{T} = \mathrm{T}'$; suppressing the quantities which cancel each other, and reducing the result, agreeably to the principles of analytical trigonometry; the following formula will be then obtained.

$$w = \left\{ \frac{1 - \text{Cos.}\,(2nT - 2mT)}{1 - \text{Cos.}\,2T} - \propto \times \frac{\text{Sin.}\,2mT - \text{Sin.}\,2nT}{\text{Sin.}\,T} + \propto^2 \right\}^{\frac{1}{2}}. \qquad \text{(XXV.)*}$$

Thus, an expression for the value of w has been obtained, which will be quite convenient for use in the field, with the table of *natural* sines and cosines, and the table of the squares and square roots of numbers subjoined to this volume. But the values of the co-ordinates x, y, and x', y', not being here computed, a new formula will be required for determining the angle P. For this purpose it will be only necessary to substitute in

$$\text{Cos. P} = \frac{x' - x - \propto}{w},$$

the values of x' and x, as obtained from (VII.) The following expression will be then obtained:

$$\text{Cos. P.} = \frac{\dfrac{\text{Sin.}\,2mT - \text{Sin.}\,2nT}{28\text{ in. }T} - \propto}{w}. \qquad \text{(XXVI.)}$$

A formula expressing the value of Cos. P. has been here selected in preference to one for the value of Sin. P, for the obvious reason that the principal term in the enumerator of (XXVI.) is had by simply dividing by 2, one of the quantities in (XXV.), whose value will always be previously known from the computation of w. But with regard to the *sign* of sine P, it may be observed that, in the case here under consideration, Sin. P will always be *positive* when $n - m$ is *positive*: and vice versa.

If in (XXV.), it be supposed that $n = m$, the result is, $w = \propto$. That is, *the distance between the two curves, in a direction*

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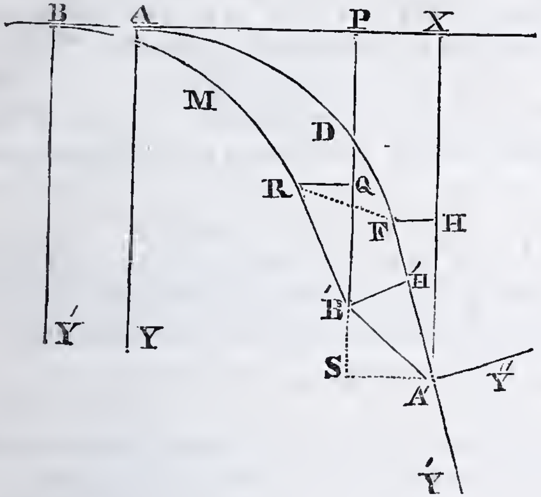
* When $2mT\ 7\ 2nT$ then $2nT - 2mT$ becomes a NEGATIVE quantity. It must however, be remembered that NEGATIVE ARCS, which are less than 90°, have POSITIVE COSINES. The quantity Sin. $2mT$ — Sin. $2nT$, will be NEGATIVE, when the latter SINE is the greatest.

The SIGN of the quantity $\propto$, is here supposed to be subject to the same CONDITIONS as in (XXIII.), and (XXIV.); and the same thing, in all cases, must be hereafter understood.

parallel to the common tangent at the origins, is always the same constant quantity $= \propto$. See art. 11.



25. Suppose ADF to represent a given curve, and BMR another proposed curve laid upon the same tangent line AX, and let $\propto$ denote the given distance AB, between their origins.—Take T and T′, to represent the given moduli of curvature; and let each curve pass into a tangent, FA′, and RB′, at the extremity of the nth and mth chain respectively. Let the number of chains contained in each tangent be denoted by v and v' respectively. It is then required to determine the distance A′ B′, between the extremities of those two tangents. And taking A, X′, A′ Y′, for a system of rectangular co-ordinate axes, coinciding with the given origin A′, and tangent line A′ F, it is proposed to investigate expressions for the values of the co-ordinates A′ H′, H′ B′, of the point B′.

The first thing which will be required in the present inquiry, is the value of each of the co-ordinates AX, XA′, and BP, PB′, of two points A′ and B′, estimated from the primitive axes AX, AY, and BX, BX′. Let those co-ordinates be represented by X, Y, and X′ Y′ respectively. The following equations will then evidently exist,

$$\begin{cases} X = x + FH \\ Y = y + A'H \end{cases};$$

but by (IV.), $\angle$ HFA' $= 2n$T, and therefore by the principles of trigonometry, FH $= v \cdot \text{Cos. } 2n\text{T}$, A'H $= v \cdot \text{Sin. } 2n\text{T}$. The following formula will therefore be the result :

$$\begin{aligned} X &= x + v \cdot \text{Cos. } 2n\text{T} \\ Y &= y + v \cdot \text{Sin. } 2n\text{T}. \end{aligned} \qquad \text{(XXVII.)}$$

And in like manner the following similar equations may be obtained :

$$\begin{aligned} X' &= x' + v' \cdot \text{Cos. } 2m\text{T}' \\ Y' &= y' + v' \cdot \text{Sin. } 2m\text{T}'. \end{aligned} \qquad \text{(XXVIII.)}$$

Now, taking W to denote the required distance A'B', its value will obviously be expressed in the following manner :

$$W = \left\{ \overline{X + \propto - Y'}|^2 + \overline{Y - Y'}|^2 \right\}^{\frac{1}{2}} \qquad \text{(XXIX.)}$$

The theorems (XXVII.) will frequently find an application in the field, as a means of investigating particular cases which will occur where tangents are concerned ; and in every case in which the line A' B' is required to be known, its value cannot be computed by any other method with more ease than by (XXIX.), a table of the squares and square roots of numbers being at hand.

It will sometimes happen that the point B' is required to be the origin of a new curve, whose modulus of curvature must be found by means of data furnished from another curve *previously computed*, or actually traced, from the origin A', and axes A' X', and A' Y'' ; and in such a case, the co-ordinates A' H', H' B', furnish the most convenient data for computing the new curvature, which was fully explained in article 22.

Put, $\propto' =$ A' H', and $\beta' =$ H' B' ; and for the sake of convenient notation, take $k = X + \propto - X' =$ S A', and $h = Y - Y' =$ S B'. It is obvious that, $\angle$ S A' H' — $2n$T, and $\angle$ S B' H' $= 180° - 2n$T ; and, therefore agreeably to a well known theorem in plane trigonometry,* a diagonal from S to H will be expressed either by

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* This theorem is sometimes wanted in the field, and it may therefore be convenient to have it expressed here, in the usual form. Take a and b to denote

$$\overline{k^2 + \alpha'^2 - 2\,k\,\alpha'.\ \text{Cos. } 2nT}|^{\frac{1}{2}},$$

or by

$$\overline{h^2 + \beta'^2 + 2h\,\beta' \cdot \text{Cos. } 2nT}|^{\frac{1}{2}};$$

these two quantities are therefore equal, and consequent'y recollecting that $\alpha'^2 + \beta'^2 = h^2 + k^2$, the result will be $\alpha'^2 - h^2 = k\,\alpha' \cdot \text{Cos. } 2nT + h \cdot (h^2 + k^2 - \alpha'^2)^{\frac{1}{2}}.\ \text{Cos. } 2nT$; that is, $\overline{\alpha'^2 - h^2}|^2 + k^2\,\alpha'^2 \cdot \text{Cos.}^2\ 2nT - 2\,k\,\alpha' \cdot (\alpha'^2 - h^2) \cdot \text{Cos. } 2nT = h^2 \cdot (h^2 + k^2 - \alpha'^2) \cdot \text{Cos.}^2\ 2nT = h^2\,k^2 \cdot \text{Cos. } 2nT - h^2 \cdot (\alpha'^2 - h^2) \cdot \text{Cos.}^2\ 2nT$; or $\overline{\alpha'^2 - h^2}|^2 - 2\,k\,\alpha' \cdot (\alpha'^2 - h^2) \cdot \text{Cos. } 2nT = -k^2 \cdot (\alpha'^2 - h^2) \cdot \text{Cos.}^2\ 2nT - h^2 \cdot (\alpha'^2 - h^2) \cdot \text{Cos.}^2\ 2nT$; that is, $\alpha'^2 - h^2 - 2\,k\,\alpha' \cdot \text{Cos. } 2nT = -(h^2 + k^2) \cdot \text{Cos.}^2\ 2nT$; and this equation is now easily reduced, by the method of quadratics, to the form, $\alpha' - k \cdot \text{Cos. } 2nT = h \cdot \text{Sin. } 2nT$. By pursuing the same method with regard to β', a similar result will be obtained; and thus the formulas which it was proposed to investigate are the following:

$$\alpha' = k \cdot \text{Cos. } 2nT + h \cdot \text{Sin. } 2nT$$
$$\beta' = k \cdot \text{Sin. } 2nT - h \cdot \text{Cos. } 2nT. \qquad \text{(XXX.)}$$

It is easy to see that the expressions just obtained might have been deduced with more facility immediately from (XXI.); but a special investigation was considered preferable. The following case may be assumed, in order to show a practical application of (XXX.).

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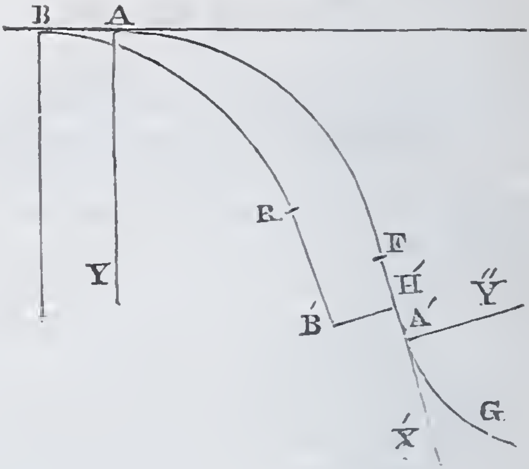
any two sides of a plane triangle, and let X represent the CONTAINED angle, and x the opposite side. Then

$$x^2 = a^2 + b^2 - 2\,a\,b\ .\ \text{Cos. } X.$$



' Example.. Let AX be a given tangent line, and A the given origin of a curve. From the origin A, and parallel to the axes AX, AY, let a system of rectangular lines be traced to a certain designated point F, selected in such a manner as to give an *integer* number of chains in the curve AF, agreeably to the method explained in art. 17 ; and let the values of T, n, x, and y, as deduced therefrom, be $T = 2° 3'$, $n = 18$ chains, $x = 13{\cdot}40$ chains, and $y = 10{\cdot}08$ chains. From the point F suppose a tangent FA' to be laid 9 chains, agreeably to the method explained in art. 16 ; and from the point A', as a new origin, and parallel to the rectangular axes A'X', A'Y'', let a second system of rectangular lines be traced, terminating in a certain designated point G, and let the resulting equations give

$$\begin{Bmatrix} x = 10{\cdot} \\ y = 10{\cdot} \end{Bmatrix}$$

chains, agreeably to art. 16. Now, having computed the modulus of curvature of A'G. and examined the direction at G, suppose it be found, in consequence of the particular situation of the ground from A' to G, to be advisable to change the origin of the curve AF to a point B, 4 chains *back* upon the tangent line AX, and from thence to lay a curve BR, from the same modulus of curvature, for a distance of 15 chains to the point R; and then a tangent RB′ for a distance of 12 chains to the point B′. It is then proposed to know what modulus of curvature will trace a curve from the tangent line RB′, and from the origin B′, passing through the same designated point G.

In such a case as the present, the co-ordinates x', y', X′, Y′, and X, Y, of the points R, B′, and A′, respectively, will most generally have been already computed in making a proper selection of the points R and B′, before any calculation is wanted with regard to the modulus of curvature of the required curve from B′ to G. But to show an example in figures, of the manner of obtaining those co-ordinates, the given data at presnet are, $T = 2° 3'$, $n = 18$, $x = 13{\cdot}40$, $y = 10{\cdot}08$, $v = 9$, $T' = 2° 3'$, $m = 15$, and $v' = 12$. Hence, $2nT = 73° 48'$, $2mT = 61° 30'$; and by (VII.),

$$x' = \frac{\text{Sin. } 61° 30'}{2 \text{ Sin. } 2° 3'} = \frac{{\cdot}87882}{{\cdot}07154} = 12{\cdot}28,$$

$$y' = \frac{1 - \text{Cos. } 61° 30'}{2 \text{ Sin. } 2° 3'} = \frac{{\cdot}52284}{{\cdot}07154} = 7{\cdot}31.$$

and by (XXVII.), $X = 13{\cdot}40; + 9 \times \text{Cos. } 73° 48' = 13{\cdot}40 + 9 \times {\cdot}279 = 13{\cdot}40 + 2.51 = 15{\cdot}91$, $Y = 10{\cdot}08 + 9 \times \text{Sin. } 73° 48' = 10{\cdot}08 + 9 \times {\cdot}960 = 10{\cdot}08 + 8{\cdot}64 = 18{\cdot}72$; and by (XXVIII.), $X' = 12{\cdot}28 + 12 \times \text{Cos. } 61° 30' = 12{\cdot}28 + 12 \times {\cdot}477 = 12{\cdot}28 + 5{\cdot}73 = 18{\cdot}01$, $Y' = 7{\cdot}31 + 12 \times \text{Sin. } 61° 30' = 7{\cdot}31 + 12 \times {\cdot}879 = 7{\cdot}31 + 10{\cdot}54 = 17{\cdot}85$. We now have $k = 15{\cdot}91 + 4{\cdot}00 - 18{\cdot}01 = 1{\cdot}90$, $h = 18{\cdot}72 - 17{\cdot}85 = 0{\cdot}87$; and by (XXX.), $\alpha' = 1{\cdot}9 \times {\cdot}279 + {\cdot}87 \times {\cdot}960 = {\cdot}53 + {\cdot}84 = 1{\cdot}37$, $\beta' = 1{\cdot}9 \times {\cdot}960 - {\cdot}87 \times {\cdot}279 = 1{\cdot}82 - {\cdot}24 = 1{\cdot}58$, which are therefore the values of the co-ordinates of the new origin B′; and thus the required modulus of curvature is readily found, by means of (XXII.), to be = 1° 56′.

It will sometimes be very convenient in the field, to determine by measurement the values of the co-ordinates, A′ H′, H′ B′, of the new origin B′, after the new line BRB′ has been traced up to the point B.′

26. Let the characters α, n, m, T, and T′, represent the same things as in the preceding article, and suppose the conditions with relation to the curves ADF, and BMR, to remain. Produce the two tangents FA′, and RB′, until they intersect each other; and take v, and v', to denote the number of chains in each respective tangent to their common point of intersection, and let z represent the angle of intersection. It is then proposed to investigate the general equations which subsist among those various quantities, in order that, when the circumstances in the field are such as to make any one of the quantities α, m, T′, v', or z, unknown, that quantity may then be eliminated, and its value obtained.

The first equation which will be required, in the investigation of any case where the *intersection* of two tangents is concerned, may be immediately deduced from (V.), and is expressed as follows:

$$2\,n\,\mathrm{T} - 2\,m\,\mathrm{T}' - z = 0. \qquad \text{(XXXI.)}$$

And thus any one of the three quantities m, T′, or z, will be made known when the *other two* are given, or assumed in such a manner as the situation of the ground may require. If, however, z be a quantity whose value is given, and fixed by particular circumstances in the field, then the value of T′ should generally be taken in such a manner as to give m an integer number, when eliminated from the equation $2\,n\,\mathrm{T} - 2\,m\,\mathrm{T}' - z = 0$.

The second subject of inquiry will now be an investigation of such equations as express the relations which exist between the quantities α, v, and v'. In the case here under consideration, it is evident from (XXIX.), that $\overline{\mathrm{X} + \alpha - \mathrm{X}'}|^2 + \overline{\mathrm{Y} - \mathrm{Y}'}|^2 = 0$; and therefore, agreeably to the principles of algebra, $\mathrm{X} + \alpha - \mathrm{X}' = 0$, and $\mathrm{Y} - \mathrm{Y}' = 0$; that is, $\alpha + x + v \cdot \mathrm{Cos.}\ 2\,n\,\mathrm{T} - x' - v' \cdot \mathrm{Cos.}\ 2\,m\,\mathrm{T}' = 0$, and $y + v \cdot \mathrm{Sin.}\ 2\,n\,\mathrm{T} - y' - v' \cdot \mathrm{Sin.}\ 2\,m\,\mathrm{T}' = 0$. From the last of these two equations, let the value of v' be obtained, and substituted in the first. The result is, $\alpha \cdot \mathrm{Sin.}\ 2\,m\,\mathrm{T}' - v \cdot \mathrm{Sin.}\ z + \overline{x - x'} \cdot \mathrm{Sin.}\ 2\,m\,\mathrm{T}' - \overline{y - y'} \cdot \mathrm{Cos.}\ 2\,m\,\mathrm{T} = 0$; and in like manner, α

$\cdot$ Sin. $2\,n\,\mathrm{T} - v' \cdot$ Sin. $z + \overline{x - x'} \cdot$ Sin. $2\,n\,\mathrm{T} - \overline{y - y'} \cdot$ Cos. $2\,n\,\mathrm{T} = 0$. But by (VII.),

$$x - x' = \frac{\text{Sin. } 2\,n\,\mathrm{T}}{2\text{ Sin. }\mathrm{T}} - \frac{\text{Sin } 2\,m\,\mathrm{T}'}{2\text{ Sin. }\mathrm{T}'},$$

and,

$$y - y' = \frac{1 - \text{Cos. } 2\,n\,\mathrm{T}}{2\text{ Sin. }\mathrm{T}} - \frac{1 - \text{Cos. } 2\,m\,\mathrm{T}'}{2\text{ Sin. }\mathrm{T}'};$$

which values being substituted in the last equations, and the obvious reductions made, the two following equations will result:

$$\propto \cdot \text{Sin. } 2\,m\,\mathrm{T}' - v \cdot \text{Sin. } z + \frac{\text{Cos. } z - \text{Cos. } 2\,m\,\mathrm{T}'}{2\text{ Sin. }\mathrm{T}} - \frac{1 - \text{Cos. } 2\,m\,\mathrm{T}'}{2\text{ Sin. }\mathrm{T}'} = 0\,;$$

$$\propto \cdot \text{Sin } 2\,n\,\mathrm{T} - v' \cdot \text{Sin } z - \frac{\text{Cos. } z - \text{Cos. } 2\,n\,\mathrm{T}}{2\text{ Sin. }\mathrm{T}'} + \frac{1 - \text{Cos. } 2\,n\,\mathrm{T}}{2\text{ Sin }\mathrm{T}} = 0\,.$$

(XXXII.)

Such is the second system of equations which will be required in the field.* The following cases may be given to illustrate their application:

Case I. When v, z, and T′, are *given*, to find $\propto$.

Here, by an evident transposition of the first of the equations (XXXII.), the following formula is obtained:

$$\propto = \frac{v \cdot \text{Sin. } z + \dfrac{1 - \text{Cos. } 2\,m\,\mathrm{T}'}{2\text{ Sin. }\mathrm{T}'} - \dfrac{\text{Cos. } z - \text{Cos. } 2\,m\,\mathrm{T}'}{2\text{ Sin. }\mathrm{T}}}{\text{Sin. } 2\,m\,\mathrm{T}'}.$$

(XXXIII.)

And the quantity $\propto$ thus becomes known; for the value of m may be obtained from (XXXI.). In every instance in which this case will occur in practice, such a value may be selected for T′, as will give m an integer number, without doing any injury to the line.

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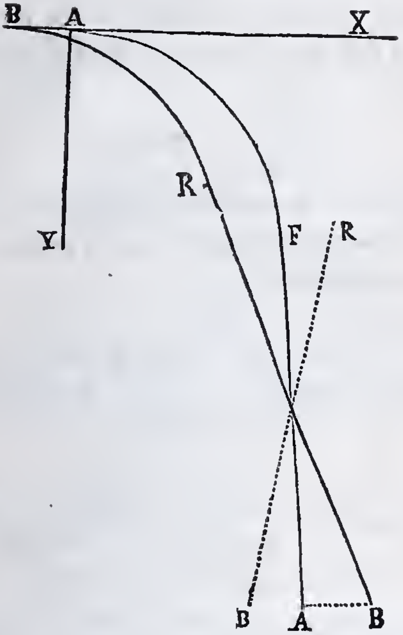
* In order to avoid misapprehension and error, particular attention must be paid to the sign of the angle z, which in all cases where 2 m T′ exceeds 2 n T, is to be made negative; or, the angle z is to be accounted negative, when the tangent v′ is more inclined than v, to the common tangent at the origins.

Example 1. Let AX be a given tangent line, and A a given point therein, selected for the origin of a curve. By means of a system of rectangular lines, traced from the origin A, and parallel to the axes AX, AY, let a certain point F be designated, as the situation of the ground may seem to require, and in such manner as to coincide with the extremity of the 25th chain of a curve AF, whose modulus of curvature is 1° 30′, agreeably to the method explained, art. 17. From the point F, let a tangent FA′ be traced 60 chains, agreeably to the principles given in art. 16 ; and from the point A′ trace the rectangular ordinate A′ B′, 2·5 chains to a point B′, selected in consequence of the particular situation of the ground. Now, suppose S to be a point in the tangent line FA′, 30 chains from F, through which the peculiar situation of the ground renders it desirable that a new tangent RSB′ should be laid. It is then proposed to determine the position of such a point B, in the primitive tangent AX, as will be the proper origin of a new curve BR, passing into the proposed tangent line RSB′.



It is here supposed, as the figure indicates, that the line **RSB′** has an inclination, less than the line FSA′ to the common tangent AX; and consequently, in this instance, the angle z will be *positive*, and expressed by the angle A′ S B′. By plane trigonometry,

$$\text{Tan. } z = \frac{A'B'}{A'S} = \frac{2\cdot5}{30} = \cdot08333\,;$$

or, $z = 4^\circ\,46'$. Hence, by (XXXI.), $2\,m\,T' = 2\,n\,T - z = 75^\circ - 4^\circ\,46' = 70^\circ\,14'$; or, $m\,T' = 35^\circ\,7'$. Now, the values of the quantities m and T' may be taken in any arbitrary manner, provided the equation $m\,T' = 35^\circ\,7'$ be satisfied. The peculiar situation of the ground, between the points A and F, must therefore *decide* the values of m and T'. The object should be to make the new curve BR as *long* as the *limits of expense* will allow; for in the same proportion as that curve is made longer, it will also be made of less abrupt curvature; but it will then diverge farther to the right of the first curve AF. Let it be supposed that the new curve BR may have a length of 20 chains; then,

$$T' = \frac{35^\circ\ 7'}{20} = 1^\circ\ 45\tfrac{1}{3}',$$

which will therefore be the modulus of curvature of the new curve BR. And now, to find the necessary position of the origin B, we have, by (XXXIII.),

$$\propto = \frac{30 \times \text{Sin. } 4^\circ\ 46' + \dfrac{1 - \text{Cos. } 70^\circ\ 14'}{2\ \text{Sin. } 1^\circ\ 45\frac{1}{3}'} - \dfrac{\text{Cos. } 4^\circ\ 40' - \text{Cos. } 70^\circ\ 14'}{2\ \text{Sin. } 1^\circ\ 30'}}{\text{Sin. } 70^\circ\ 14'}$$

That is,

$$\propto = \frac{30 \times \cdot 08310 + \dfrac{\cdot 66181}{\cdot 06128} - \dfrac{\cdot 99654 - \cdot 33819}{\cdot 05236}}{\cdot 94108}$$

$$= \frac{2{\cdot}493 + 10{\cdot}800 - 12{\cdot}573}{\cdot 941} = \frac{\cdot 720}{\cdot 941} = \cdot 765.$$

Hence, measure the distance A B = ·765 of a chain, *back* upon the tangent line AX, and the required *origin* of the new curve BR will be obtained.

Example 2. Suppose the same data to remain as in the preceding instance, with the exception only that the position of the required new tangent R′ S B″, is reversed; that is, let the line R′ S B″ have an inclination exceeding that of the line FSA′, to the common tangent AX.

Here the angle z becomes *negative*, and therefore, by (XXXI.), $2\, m\, T' = 2\, n\, T - z = 75^\circ + 4^\circ\ 46' = 79^\circ\ 46'$; or, $m\, T' = 39^\circ\ 53'$. And hence, if it be supposed that the ground between the points A and F be such as to admit a new curve 40 chains in length, then

$$T' = \frac{39^\circ\ 53'}{40} = 0^\circ\ 59\tfrac{4}{5}' =$$

modulus of curvature of the new curve. Now, recollecting that Sin. z becomes *negative*, and Cos. z remains *positive*, we have

$$\propto = -30 \times \text{Sin.}\ 4^\circ\ 46' + \frac{\dfrac{1 - \text{Cos.}\ 79^\circ\ 46'}{2\ \text{Sin.}\ 59\frac{4}{5}'} - \dfrac{\text{Cos.}\ 4^\circ\ 46' - \text{Cos.}\ 79^\circ\ 46'}{2\ \text{Sin.}\ 1^\circ\ 30'}}{\text{Sin.}\ 79^\circ\ 46'}$$

$$= \frac{-2.493 + \dfrac{\cdot 82234}{\cdot 03480} - \dfrac{\cdot 99654 - \cdot 17766}{\cdot 05236}}{\cdot 98409}$$

$$= \frac{-2\cdot 493 + 23\cdot 630 - 15.637}{\cdot 98409} = +\frac{5\cdot 500}{\cdot 984} = 5\cdot 59.$$

Hence, the new origin is 5·59 chains *back* upon the tangent line AX; and consequently, in this instance, the new *curve* will *intersect* the first curve AF.

Case II. When $\propto$, v, and z, are given; to find T'.

Let the quantity $2\,m\,T'$ be represented by D; and the following expression will be immediately derived from (XXXI.):

$$D = 2\,n\,T - z. \qquad \text{(XXXIV.)}$$

The value of D will be thus made known; and by an obvious transposition of the first equation (XXXII.), the following formula will then obtain:

$$\text{Sin. } T' = \frac{1 - \text{Cos. } D}{2 \propto \cdot \text{Sin. } D - 2v \cdot \text{Sin } z + \dfrac{\text{Cos. } z - \text{Cos. } D}{\text{Sin. } T}}. \qquad \text{(XXXV.)}$$

Having computed the value of T', the quantity m, which denotes the number of chains contained in the new curve, will be made known by the formula,

$$m = \frac{D}{2\,T'}.$$

This will, however, be accurately true, only when m is an *integer* number, for reasons already explained in art. 5; but the formula (XXXV.) is nevertheless, obviously rigorous. When the formula

$$\frac{D}{2\,T'},$$

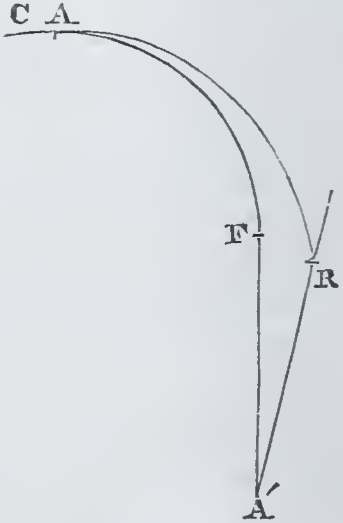
does not express an *integer* number, it will be, most generally, convenient in practice to take for the value of m the nearest integer number *greater* than

$$\frac{D}{2\,T'};$$

and then, after tracing every chain in the new curve *except the last*, let that last chain be laid from a modulus of curvature expressed by the formula,

$$\tfrac{1}{2}D - T' \times \overline{m - 1};$$

which will restore the proper *direction* to the new tangent, agreeably to principles evident enough from (IV.), and it will not vary, laterally, any material distance from the required position.



Example. Let CADF be a curve whose modulus of curvature is 1° 30′; and let A be a station in that curve 25 chains from the extreme station F. From F, suppose a tangent FA′ to be laid 30 chains to a point A′. It is then proposed to determine the necessary change of curvature at the station A, in order to trace a new curve AR, such as to pass into a new proposed tangent RA′, intersecting the former at the point A′, and whose inclination to the common tangent at the station A, *exceeds* that of the tangent FA, by 4° 46′.

Here, $T = 1° \, 30'$, $n = 25$, $v = 30$, $\propto\, = 0$, and $z = -\, 4° \, 46'$; therefore by (XXXIV.), $D = 2\, n\, T - z = 75° + 4° \, 46' = 79° \, 46'$. Hence, by (XXXV.),

$$\text{Sin. } T' = \frac{\cdot 82234}{+\, 60 \times \cdot 08310 + \dfrac{\cdot 99654 - \cdot 17766}{\cdot 02618}}$$

$$= \frac{\cdot 82234}{4{\cdot}986 + 31{\cdot}278} = \frac{\cdot 82234}{36{\cdot}264} = \cdot 02267\, ;$$

or, $T' = 1° \, 18'$ = modulus of curvature necessary to trace the required curve AR, agreeably to the principles explained in art. 9. Now, in this instance,

$$\frac{D}{2\, T'} = \frac{79° \, 46'}{2° \, 36'} = \frac{79{\cdot}767°}{2{\cdot}6°} = 30{\cdot}68\, ;$$

which is not an *integer* number, and the new curve AR must therefore be made to consist of 31 chains, of which the first 30 will be laid agreeably to the modulus of curvature 1° 18′. And then,

$\frac{1}{2}D - T' \times \overline{m - 1} = 39° \, 53' - 1° \, 18' \times \overline{31 - 1} = 39° \, 53' - 39° \, 0' = 0° \, 53'$ = necessray modulus of curvature for the 31st chain.

27. The preceding articles, with their obvious combinations, embrace all the cases which can occur in the field, by a method of computation rigorously accurate, and of convenient application. But before closing this part of the subject, the following example will be here given, as a general illustration of the method to be pursued when an alteration is proposed in a line after the completion of a location.

Let D A F A′ G L represent a located line having the following character : D A, a tangent ; A F, a curve, modulus of curvature 2° 3′, and length 18 chains ; F A′, a tangent, length 9 chains ; A′ G L, a curve, modulus of curvature of the part A′ G, 2° 52′, and length of the same part 15 chains. Now, it is proposed to remove the origin A 4 chains *back* upon the tangent line D A, to a station B ; and to lay a curve B R 15 chains, with the same modulus of curvature, 2° 3′ ; and it is required to know what would be the direct distance R Q between the two curves from the point R. And if a tangent R X′ be laid from the point R, it is also required to know the proper position in that tangent line, for the orign B′, and the necessary modulus of curvature, in order to trace a new curve B′ G in such a manner as to pass again into the original line G L with a common tangent at the station G.

Because B R = 15 chains, and B A = 4 chains, the difference is 11 chains, and the curve A Q will therefore obviously contain more than 11 chains when R Q is normal to the tangent R X′. Hence, taking the station Q at the extremity of 13 chains, the data for computing the line R Q by means of (XXV.), will be, $T' = 2° \, 3'$, $n = 13$, $m = 15$, and $\propto = +\,4$; or $2nT = 53° \, 18'$, and $2mT = 61° \, 30'$; and therefore

$$w = \left\{ \frac{\cdot 01022}{\cdot 00256} - 4 \times \frac{\cdot 87882 - \cdot 80178}{\cdot 03577} + 16 \right\}^{\frac{1}{2}}$$

$= (4 \cdot 00 - 4 \times 2 \cdot 153 + 16)^{\frac{1}{2}} = \sqrt{11 \cdot 388} = 3 \cdot 38$ chains nearly $=$ R Q. And by (XXVI.),

$$\text{Cos. P} = \frac{1 \cdot 076 - 4}{3 \cdot 38} = - \cdot 8651 \, ;$$

that is $P = 210° \, 6'$. Hence, the angle Q R X′ will evidently be expressed by $2mT + 210° \, 6' - 180° = 91° \, 36'$; which being nearly a right angle, proves that 13 integer chains in the curve A Q, corresponds most nearly with a true normal line Q R. The length of the line Q R furnishes the data by which to judge of the situation of the ground at the point R.

The origin and curvature of the new curve B′ G, will now be investigated by means of (XVII.) and (XVIII.) Let any arbitrary point K be selected for the origin of a system of rectangular axes K X′, K Y′ ; and compute the values of the co-ordinates A′ H′, H′ K, of the origin K, taken with reference to the rectangular axes which have their origin at A′.

In such a case as the present, it would be most convenient to take the point K coinciding with the station R; but in order to retain the results given in a former example, take R K = 12 chains; and the values of the required co-ordinates will then be A′ H′ = + 1·37 chains, and H′ K = + 1·58 chains.

The co-ordinates of the point G. taken with reference to the axes K X′, K Y′, may now be computed by means of (XXI.), after the co-ordinates of the same point, taken with reference to the axes at A′, have been determined by (VII.) To find these latter co-ordinates, the given data are, T = 2° 52′, and $n = 15$; or, $2n\mathrm{T} = 86°$. Hence, by (VII.),

$$x = \frac{\cdot 99756}{\cdot 10002} = 9\cdot 98 \text{ chains}, \; y = \frac{\cdot 93024}{\cdot 10002} = 9\cdot 30 \text{ chains}.$$

Now, the data necessary for computing the new co-ordinates by means of (XXI.), are $\alpha = +\,1\cdot 37$, $\beta = +\,1\cdot 58$, $z = +\,12°\,18'$, $x = 9\cdot 98$, and $y = 9\cdot 3$. Hence, $x' = 10\cdot 88 \times \cdot 2130 + 11\cdot 35 \times \cdot 9770 = 2\cdot 32 + 11\cdot 09 = 13\cdot 41$ chains, $y = 10\cdot 88 \times \cdot 977 - 11\cdot 35 \times \cdot 213 = 10\cdot 63 - 2\cdot 42 = 8\cdot 21$ chains. The inclination of a tangent at G, to the tangent F A′, is expressed by 30 × 2° 52′ = 86°; and therefore the inclinations of a tangent at G, to the tangent R X′, is expressed by 86° — 12° 18′ = 73° 42′. The given data by which to compute the position of the new origin B′ by means of (XVIII.), will therefore be X = 13·41, Y = 8·21, and D = 73° 42′; that is $\alpha = 8\cdot 21 \times 1\cdot 334 - 13\cdot 41 = 10\cdot 95 - 13\cdot 41 = -\,2\cdot 46$. The required origin B′, of the new curve B′ G, will therefore be situated 2·46 chains in *advance* of the selected point K; that is, the length of the new tangent R B′ is 14·46 chains. By (XVII.),

$$\text{Sin. T} = \frac{\cdot 71933}{16\cdot 42} = \cdot 04381\,;$$

or, T = 2° 30¾′ = modulus of curvature of the required new curve B′ G.

SCHOLIUM.

When the curves under consideration are *long*, and also embrace portions of circumferences greatly exceeding 20° or 30°, the trigonometrical formulas investigated in the foregoing articles become of the utmost importance in the field. But there are several well known approximative formulas which will sometimes be found useful auxilliaries in the first location of a line where short and frequent curves are introduced; and they will also very often be convenient when certain alterations are proposed in a known line, and a new line is required to be selected by computation. It is therefore the design of the remaining part of the present inquiry to show how those approximative results may be easily deduced from the rigorous formulas already given.

28. By means of (VI.), we have

$$y = \frac{\text{Sin. }^2 nT}{\text{Sin. T}};$$

and therefore the quantity $n^2 \times$ Sin. T will always *exceed* the value of y. Hence $n^2 \times$ Sin. T will express the value of some ordinate *oblique* to the axis of x.

Let this *oblique ordinate* be denoted by K, and the following expression will obtain:

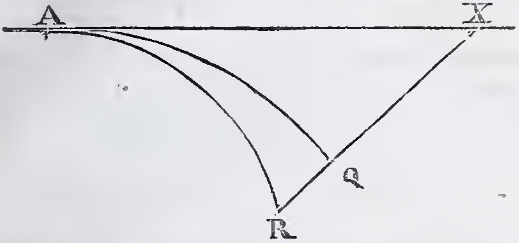
$$K = n^2 \cdot \text{Sin. T}. \qquad \text{(XXXVI.)}$$

But by putting $q = 0{\cdot}01745$, which is the length of an arc of one *degree*, to a radius unity, and supposing the angle T to be measured *in degrees, and decimal parts of a degree*, the result will be, that $q \times T =$ Sin. T very nearly. The value of the oblique ordinate K will therefore be nearly expressed as follows:

$$K = qn^2\, T. \qquad \text{(XXXVII.)}$$

29. Suppose T and T′ to denote the respective moduli of curvatures *in degrees* of two curves A Q and A R, which are laid from the same point A, and upon the same tangent line A X; and let each curve contain an equal number of chains rep-

resented by n. It is proposed to find an approximate value for the distance Q R.



Let Q X and R X be the two oblique ordinates whose values are expressed by qn^2 T, and qn^2 T′, respectively. If, then those two ordinates be supposed to coincide with each other, which will not be far from the truth, then their difference in length must express the required distance Q R. Hence taking w to represent the line Q R, the following formula is evidently the result :

$$w = qn^2 \cdot (T - T'). \qquad \text{(XXXVIII.)}$$

The formula thus obtained is an exceedingly near approximation of the true length of the line Q R, when the curves A Q and A R do not contain more than 20° or 30° each ; but its application in the field is confined to that particular case only in which those two curves contain each the same number of chains. And although the length of the line w, as thus computed, will not differ so much from the true quantity as to produce a material inconvenience in the location of a line in a slightly undulating country, even when the two curves embrace portions of circumferences greatly exceeding 20° or 30°, yet in such cases, if the result were even rigorously true, a knowledge of the *length* of the line w will be of little use in the field when the *obliquity* of that line remains unknown. The formulas (XXIII.) and (XXIV.) must, therefore, in such a case, be resorted to, in order to discover the true position of one of the curves when the other is given ; but when those two curves each embrace only a small part of the whole circumference, it will be sufficient in practice to measure the line w as a normal to either curve, in which case the expression (XXXVIII.) will be highly useful.

It is sometimes required to find the modulus of curvature of one of the curves, when the other curve and the line w are given data; in which case the following formula obtains:

$$T' = T \pm \frac{w}{qn^2} \qquad \text{(XXXIX.)}$$

This expression is immediately derived from a transposition of the one preceeding.

Three examples will be now given, the first of which will be a case where the formula (XXXVIII.) is not at all applicable; the second will explain a case where that formula can not be used alone with advantage; and the third will show an instance in which the same formula will be very useful in the field.

Example 1. Let a curve whose modulus of curvature is 2° 45′, be traced 20 chains of 100 feet each, from a given point and from a given tangent line. Now it is proposed to lay another curve from the same origin, and from the same tangent line, and whose modulus of curvature is 3°. How far would this latter curve pass from the extremity of the former?

Here, by (XXXVIII.), $w = \cdot 01745 \times 400 \times \frac{1}{4}^{\circ} = 1\cdot 75$ chains $= 175$ feet $=$ the required distance as determined by the approximative method. But this result is too far from the truth to be of any practical utility, as will appear from the following acurate calculation. Taking the extremity of the 19th chain for the termination of the new curve, the given data in (XXIII.) are, $n = 20$ $m = 19$, $T = 2^{\circ}\,45'$, $T' = 3^{\circ}$, and $\propto\, = 0$; that is,

$$x = \frac{\text{Sin. } 110^{\circ}}{2 \text{ Sin. } 2^{\circ}\,45'} = \frac{\cdot 93969}{\cdot 0960} = 9\cdot 79\,;$$

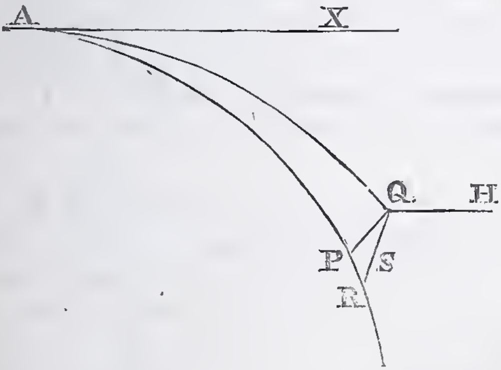
$$y = \frac{1 - \text{Cos. } 110^{\circ}}{2 \text{ Sin. } 2^{\circ}\,45'} = \frac{1 + \cdot 34202}{\cdot 0960} = \frac{1\cdot 34202}{\cdot 0960} = 13\cdot 98\,;$$

$$x' = \frac{\text{Sin. } 114^\circ}{2 \text{ Sin. } 3^\circ} = \frac{\cdot 91355}{\cdot 1047} = 8\cdot 73\,;$$

$$y' = \frac{1 - \text{Cos. } 114^\circ}{2 \text{ Sin. } 3^\circ} = \frac{1 + \cdot 40674}{\cdot 1047} = \frac{1\cdot 40674}{\cdot 1047} = 13\cdot 44\,;$$

or, $x - x' = 1\cdot 06$, and, $y - y' = 0\cdot 54$. Hence $w = (\overline{1.06}|^2 + \overline{\cdot 54}|^2)^{\frac{1}{2}} = (1\cdot 1236 + \cdot 2916)^{\frac{1}{2}} = \sqrt{1\cdot 415} = 1\cdot 19$ chains = 119 feet = true distance from the extremity of the given curve, to the station at the termination of the 19th chain in the proposed curve.

Example 2. Let A R represent the curve of a graded roadway, and suppose A Q to be an approximate curve, traced 10 chains, of 100 feet each, with a modulus of curvature of 2° 45′. From the extremity at Q, suppose a normal line Q P to be measured 40 feet to the centre of the roadway at P. It is proposed to show a method of ascertaining the modulus of curvature which will trace the curve A R.



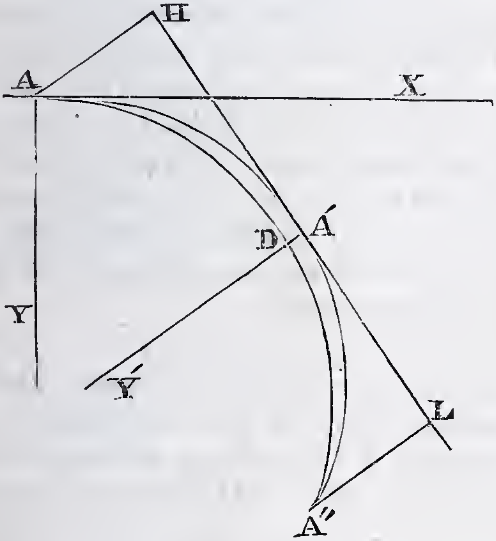
In order to obtain an approximate value for the new modulus of curvature by means of (XXXIX.), the given data are, $T = 2^\circ\ 45'$, $n = 10$, and $w = \cdot 4$; that is,

$$T' = 2^\circ\ 45' + \frac{\cdot 4}{\cdot 0175 \times 100} = 2^\circ\ 45' + \frac{\cdot 4}{1\cdot 75} = 2^\circ\ 45' + \cdot 229^\circ = 2^\circ\ 45' + 13\tfrac{3}{4}'. = 2^\circ\ 58\tfrac{3}{4}'.$$

Now, if a new curve were to be commenced at the origin A, and traced agreeably to the modulus of curvature $2^\circ\ 58\frac{3}{4}'$, it would not touch the point P by a deviation of about 3 feet, as an accurate calculation will show, and a variation of even 6 inches from the true centre of a graded surface is frequently a matter of some consequence. If a small error be made in the modulus of curvature at the commencement of a curve, it is not a proper remedy, after the line begins to deviate a material quantity from the true centre, to make a new change of curvature merely to correct the former error; for such a method of operation not only shows a great want of skill, but it also multiplies the difficulties when the road at any future time may want adjustment. When, therefore, the approximate result, $2^\circ\ 58\frac{3}{4}'$, has been obtained, the next thing required to be done, by means of an instrument placed at Q, is to ascertain the true position of a point S, where the extremity of the 10th chain in the new curve would be situated, without actually tracing that curve upon the ground. The length and position of the line Q S may be correctly computed by means of (XXIII.), and (XXIV.); and then having measured the amount of error S R, to the centre of the roadway, the modulus of curvature required may be determined by means of (XXXIX.). For the given data will then be $T = 2^\circ\ 58\frac{3}{4}'$, $n = 10$, and $w =$ measured distance S R.

It will be here easily observed that there are two sources of error in the first result obtained from (XXXIX.). The greatest part of this error arises from the want of coincidence in the directions of the lines Q P and Q S; for an accurate calculation will show that in the present example, a difference of $13\frac{3}{4}'$ in the moduli of curvatures corresponds accurately with a distance of 39 feet between the two extremities of the curves, which agrees with the distance supposed in the first calculation, within one foot. But owing to the obliquity of the line Q S, a new curve laid from the origin A, agreeably to the modulus of curvature $2^\circ\ 58\frac{3}{4}'$, would intersect the line Q P at about 37 feet from the point Q, and therefore pass 3 feet from the true centre of the roadway.

Example 3. Suppose A A′ and A′ A″ to represent two different curves, connected together upon a common tangent at the point A′, and selected in the field by tracing systems of rectangular lines agreeably to the method of co-ordinate axes as explained in art. 17; and let the *numerical values* of all the quantities remain as in the example given for that article. Now, in the place of these two curves, it is proposed to lay one continuous curvature from A to A″, if the ground about the intermediate point A′ will permit such an alteration; and it is therefore required to know how far the new proposed curve A D A″ would pass from the point A′.



It is very evident that the co-ordinates A″ H, H A, of th point A, taken with reference to the axes A′ X′, A′ Y′, are re spectively equal to the co-ordinates x, y, of the A′, taken wit reference to the axes A X, A Y. Hence, the given data fo computing the modulus of curvature of the proposed curve A D A″, by means of (XXII.), are the following : $\alpha =$ A′ H $= 20{\cdot}21$ chains, $\beta = -$ H A $= -\ 12{\cdot}47$ chains, x 17·1 chains, $y = 10{\cdot}24$ chains, and $z = -\ 63^\circ\ 20'$. Therefor

$$\text{Sin. } T' = \frac{-2{\cdot}23 \times \text{Cos. } 63^\circ\, 20' + 37{\cdot}39 \times \text{Sin. } 63^\circ\, 20'}{\overline{-2{\cdot}23}|^2 + \overline{37{\cdot}39}|^2}$$

$$= \frac{32{\cdot}412}{1403} = {\cdot}02310\,;$$

or, T′ = 1° 20′ = modulus of curvature which would trace the continuous curve A D A″.

The data for computing the distance A′ D by means of (XXXVIII.), will now be, $n = 25$, T = 1° 16′ and T′ = 1° 20′ ; that is,

$$w = {\cdot}0175 \times 625 + \frac{4}{60} = 0{\cdot}75 =$$

the required distance A′ D nearly, expressed in chains.

30 When two curves, having their respective moduli of curvatures represented in degrees by T and T′, are laid from different origins upon the same tangent line, let $\propto$ denote the number of chains between the origins. If, then, the curve T be continued n chains, and the curve T′ be supposed to contain $n \pm \alpha$ chains, it is evident that the following formula will obtain fo a reason similar to that in the preceding article, the quantity w denoting the same line as before :

$$w = qn^2\, T - q \cdot \overline{n \pm \propto}|^2 \cdot T'. \qquad \text{(XL.)}$$

The expression thus obtained will be frequently useful in the field when *short curves* only are under consideration ; and if it be supposed that $w = 0$, a formula may obviously be easily deduced which will correspond to (XIX.) by an approximation. For then

$$n^2\, T = \overline{n \pm \propto}|^2 \cdot T'\,;$$

that is,

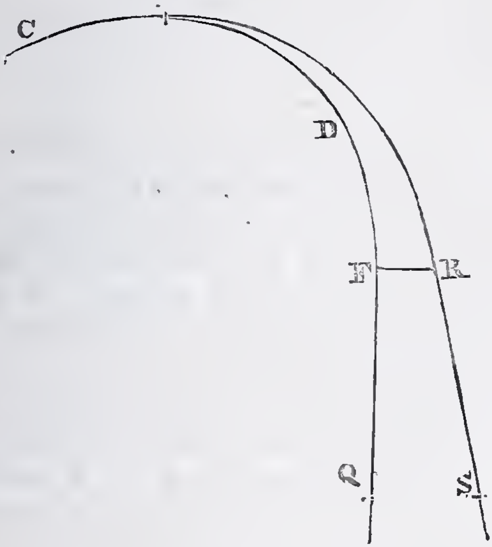
$$T' = T \times \overline{\frac{n}{n \pm \propto}}\Big|^2. \qquad \text{(XLI.)}$$

This expression is remarkably simple, and will be very convenient for short curves.

31. Let C A D F represent a given curve, whose modulus of curvature is denoted in degrees by T, and let it pass into a

tangent F Q at any given station F. Let also F R represent any small distance immediately on the right or left of the station F, and R S a straight line, whose inclination to the line F Q is denoted by a given quantity $z°$. It is then proposed to determine a point A in the given curve C A D F, and the requisite change of curvature at A, in order to lay the curve A D F into the position A R, and the tangent F Q into the position R S.

Take w to represent the small given distance F R, and let n denote the required number of chains from A to F, and T′ the required modulus of curvature of the new curve A R.

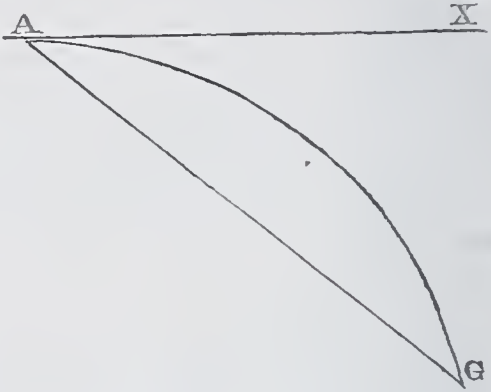


The two equations, $z = 2n \cdot (T - T')$, and $w = qn^2 \cdot (T - T')$, are furnished by means of (V.) and (XXXVIII.) respectively; and therefore, by eliminating n and $T - T'$, the following expressions will result:

$$n = \frac{2w}{qz}$$

$$T - T' = \frac{qz^2}{4w} \qquad \text{(XLII.)}$$

These formulas will be very convenient in the field when two short curves in reversion are to be connected by means of an intervening tangent common to both curves.



32. Suppose A X to represent a given tangent line, and A a given point therein, which is designed for the origin of a certain required curve to be laid passing through a given point G.—Suppose, also, that the point G is visible from the origin A, and that the approximate distance A G is known. It is then proposed to find the necessary modulus of curvature, in order to trace the required curve A G.

By means of an instrument placed at the origin A, let the angle X A G be measured, and let its value be denoted by D′. It evidently appears from (II.) that the angle D′ will contain the modulus of curvature as often as the curve A G contains chains. Hence,

$$T = \frac{D'}{n} \qquad \text{(XLIII.)}.$$

The expression thus obtained is extremely simple, and for short curves will be highly useful in the field.